

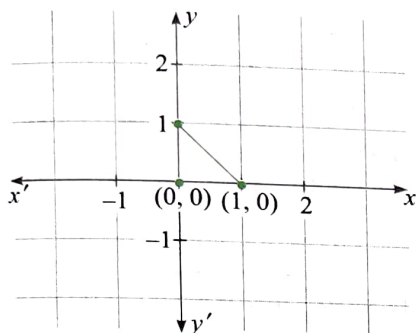
Solutions

Chapter 1

Concept Application Exercises

Exercise 1.1

1. Clearly, for minimum area of triangle height and base must be minimum.



$$\therefore \text{Minimum area of triangle} = \frac{1}{2} \times 1 \times 1 = \frac{1}{2} \text{ sq. units}$$

2. Given points $A(2, 3)$ and $B(7, 5)$.

Feet of perpendiculars from these points on x -axis are $C(2, 0)$ and $D(7, 0)$.

$$\therefore CD = 7 - 2 = 5 \text{ units}$$

3. Given point is $P(-2, 3)$.

Changed position of point P after said transformations are:

- (i) reflection in the line $y = x$: $(3, -2)$
- (ii) translation of 4 units to the right: $(3 + 4, -2) \equiv (7, -2)$
- (iii) translation of 5 units up: $(7, -2 + 5) \equiv (7, 3)$
- (iv) reflection in y -axis: $(-7, 3)$

4. Here, the new origin is $(2, -3)$. Then,

$$x = X + 2, y = Y - 3$$

and the given equation transforms to

$$(X + 2)^2 + 7(X + 2)(Y - 3) - 2(Y - 3)^2 + 17(X + 2) - 26(Y - 3) - 60 = 0$$

$$\text{or } X^2 + 7XY - 2Y^2 - 4 = 0$$

5. Let origin be shifted at point (h, k) without rotating the coordinate axes.

Now, we replace x by $(x + h)$ and y by $(y + k)$ in the equation of given curve.

Then the transformed equation is

$$(x + h)^2 + (y + k)^2 - 4(x + h) + 6(y + k) - 7 = 0$$

$$\Rightarrow x^2 + y^2 + x(2h - 4) + y(2k + 6) + h^2 + k^2 - 4h + 6k - 7 = 0$$

Since, this equation does not contain the terms of first degree.

$$\therefore 2h - 4 = 0 \text{ and } 2k + 6 = 0$$

$$\Rightarrow h = 2 \text{ and } k = -3$$

Hence, the point to which the origin should be shifted is $(2, -3)$.

6. Comparing the given equation with

$$ax^2 + 2hxy + by^2$$

$$\text{we get } a = 4, h = \sqrt{3}, b = 2.$$

Let θ be the angle through which the axes are to be rotated. Then,

$$\tan 2\theta = \frac{2h}{a - b}$$

$$\text{or } \tan 2\theta = \frac{2\sqrt{3}}{4 - 2} = \sqrt{3} = \tan \frac{\pi}{3}$$

$$\text{or } 2\theta = \frac{\pi}{3}, \pi + \frac{\pi}{3}$$

$$\text{or } \theta = \frac{\pi}{6}, \frac{2\pi}{3}$$

Exercise 1.2

$$\begin{aligned} 1. PQ &= \sqrt{a^2(\cos \alpha - \cos \beta)^2 + a^2(\sin \alpha - \sin \beta)^2} \\ &= a \sqrt{\sin^2 \alpha + \cos^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2\cos \alpha \cos \beta - 2\sin \alpha \sin \beta} \\ &= a \sqrt{2(1 - \cos(\alpha - \beta))} \\ &= a \sqrt{2 \times 2 \sin^2 \frac{\alpha - \beta}{2}} \\ &= 2a \sin \frac{\alpha - \beta}{2} \end{aligned}$$

2. (i) $A(-2, 2), B(8, -2), C(-4, -3)$

$$\begin{aligned} AB &= \sqrt{(8 - (-2))^2 + (-2 - 2)^2} \\ &= \sqrt{100 + 16} = 2\sqrt{29} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(8 - (-4))^2 + (-2 - (-3))^2} \\ &= \sqrt{144 + 1} = \sqrt{5} \sqrt{29} \end{aligned}$$

$$\begin{aligned} CA &= \sqrt{(-2 - (-4))^2 + (2 - (-3))^2} \\ &= \sqrt{4 + 25} = \sqrt{29} \end{aligned}$$

$$\text{Thus, } AB^2 + CA^2 = BC^2$$

So, triangle is right angled at A .

- (ii) $A(-a, -b), B(a, b), C(a^2, ab)$

$$AB = \sqrt{(2a)^2 + (2b)^2} = 2\sqrt{a^2 + b^2}$$

$$BC = \sqrt{(a^2 - a)^2 + b^2(a - 1)^2} = (a - 1)\sqrt{a^2 + b^2}$$

$$AC = \sqrt{(a^2 + a)^2 + b^2(a + 1)^2} = (a + 1)\sqrt{a^2 + b^2}$$

$$\text{Thus, } AB + BC = AC.$$

So, points are collinear.

- (iii) $A(4, 0), B(-1, -1), C(3, 5)$.

$$AB = \sqrt{(-1 - 4)^2 + (-1 - 0)^2} = \sqrt{25 + 1} = \sqrt{26}$$

$$BC = \sqrt{(3 + 1)^2 + (5 + 1)^2} = \sqrt{16 + 36} = \sqrt{52}$$

$$CA = \sqrt{(4 - 3)^2 + (0 - 5)^2} = \sqrt{1 + 25} = \sqrt{26}$$

$$\text{Thus, } AB = CA \text{ and } BC^2 = AB^2 + CA^2$$

So, triangle ABC is right angled isosceles.

3. The given points are collinear,

$$\therefore \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 0 & \sec^2 \theta \\ \operatorname{cosec}^2 \theta & 0 \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \sec^2 \theta - \operatorname{cosec}^2 \theta \sec^2 \theta + \operatorname{cosec}^2 \theta = 0$$

$$\Rightarrow \frac{1}{\cos^2 \theta} - \frac{1}{\sin^2 \theta \cos^2 \theta} + \frac{1}{\sin^2 \theta} = 0$$

$$\Rightarrow \frac{1}{\sin^2 \theta \cos^2 \theta} - \frac{1}{\sin^2 \theta \cos^2 \theta} = 0$$

Therefore, the points are collinear for all value of θ , except only

$$\theta = \frac{n\pi}{2}, n \in \mathbb{Z}.$$

$$\text{Thus, } \theta \in \mathbb{R} - \left\{ \frac{n\pi}{2}, n \in \mathbb{Z} \right\}$$

4. We have $A(2, 4)$ and $B(6, 7)$ as end points of diagonal of the hexagon.

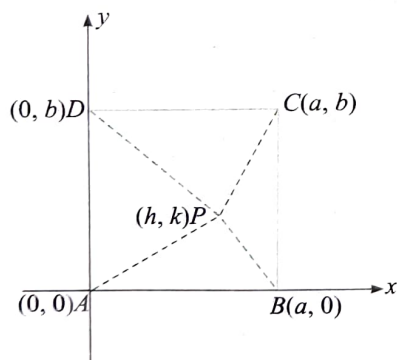
$$\therefore AB = 5$$

Thus, length of the line segment joining centre of the hexagon and one of the vertices is $\frac{5}{2}$.

$$\begin{aligned} \therefore \text{Area of hexagon} &= 6 \times \text{Area of equilateral triangle having} \\ &\quad \text{side length } \frac{5}{2} \\ &= 6 \times \frac{\sqrt{3}}{4} \times \frac{25}{4} = \frac{75\sqrt{3}}{8} \end{aligned}$$

5. Let $A \equiv (0, 0)$, $B \equiv (a, 0)$, $C \equiv (a, b)$ and $D \equiv (0, b)$

Let point P be (h, k) .

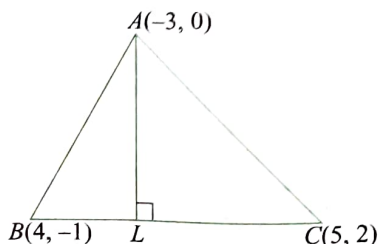


$$\begin{aligned} \therefore PA^2 + PC^2 &= (h-0)^2 + (k-0)^2 + (h-a)^2 + (k-b)^2 \\ &= 2h^2 + 2k^2 - 2ah - 2bk + a^2 + b^2 \end{aligned}$$

$$\begin{aligned} \text{Also, } PB^2 + PD^2 &= (h-a)^2 + (k-0)^2 + (h-0)^2 + (k-b)^2 \\ &= 2h^2 + 2k^2 - 2ah - 2bk + a^2 + b^2 \end{aligned}$$

$$\text{Thus, } PA^2 + PC^2 = PB^2 + PD^2$$

- 6.



In the figure, AL is altitude.

$$\text{Area of triangle} = \frac{1}{2}[-3(-1-2) + 4(2-0) + 5(0+1)] = 11$$

$$\text{Also, } BC = \sqrt{(5-4)^2 + (2+1)^2} = \sqrt{1+9} = \sqrt{10}$$

$$\text{Now, area of triangle} = \frac{1}{2} \times AL \times BC$$

$$\therefore AL = \frac{2 \times 11}{\sqrt{10}} = \frac{22}{\sqrt{10}}$$

7. The required area is

$$\begin{aligned} \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 7 & 21 \\ 12 & 2 \\ 7 & -3 \\ 0 & -3 \\ 1 & 1 \end{vmatrix} &= \frac{1}{2}[(21 + 14 - 36 - 21) - (7 + 252 + 14 - 3)] \\ &= 146 \text{ sq. units} \end{aligned}$$

$$8. \frac{\text{Area of } \triangle DBC}{\text{Area of } \triangle ABC} = \frac{1}{2}$$

$$\text{or } \frac{\frac{1}{2}[x(5+2) - 3(-2-2x) + 4(2x-5)]}{\frac{1}{2}[6(5+2) - 3(-2-3) + 4(3-5)]} = \frac{1}{2}$$

$$\text{or } \frac{7x+6+6x+8x-20}{42+15-8} = \frac{1}{2}$$

$$\text{or } \frac{3x-2}{7} = \frac{1}{2} \quad \text{or } x = \frac{11}{6}$$

Exercise 1.3

1. Let $A \equiv (a, b)$ and $B \equiv (c, d)$.

$P(3, 2)$ divides AB internally in the ratio $3 : 2$.

$$\therefore (3, 2) \equiv \left(\frac{3c+2a}{3+2}, \frac{3d+2b}{3+2} \right)$$

$$\therefore 3c+2a=15 \quad (1)$$

$$\text{and } 3d+2b=10 \quad (2)$$

$P(-2, 3)$ divides AB externally in the ratio $4 : 3$.

$$\therefore (-2, 3) \equiv \left(\frac{4c-3a}{4-3}, \frac{4d-3b}{4-3} \right)$$

$$\therefore 4c-3a=-2 \quad (3)$$

$$\text{and } 4d-3b=3 \quad (4)$$

Solving (1) and (3), we get

$$17c=41 \text{ and } 17a=66$$

Solving (2) and (4), we get

$$17d=36 \text{ and } 17b=31$$

$$\therefore A \equiv \left(\frac{66}{17}, \frac{31}{17} \right) \text{ and } B \equiv \left(\frac{41}{17}, \frac{36}{17} \right)$$

2. The midpoints of the diagonals must be the same. Therefore,

$$\frac{x-2}{2} = \frac{-3+3}{2}$$

$$\text{or } x=2$$

$$\text{and } \frac{-1+3}{2} = \frac{-2+y}{2}$$

$$\text{or } y=4$$

3. Let the vertices of triangle be (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) . Then,

$$\frac{x_1+x_2}{2} = 2, \frac{x_2+x_3}{2} = -1, \frac{x_3+x_1}{2} = 4$$

Solving, we get $x_1 = 7$, $x_2 = -3$, and $x_3 = 1$.

Similarly, $y_1 = 9$, $y_2 = -7$ and $y_3 = 1$.

4. $A(b \cos \alpha, b \sin \alpha)$ $B(a \cos \beta, a \sin \beta)$ $M(x, y)$

$$\frac{AM}{BM} = \frac{b}{a}$$

$$\therefore M\left(\frac{ab \cos \alpha - ab \cos \beta}{(a-b)}, \frac{ab \sin \alpha - ab \sin \beta}{(a-b)}\right) \equiv M(x, y)$$

$$\Rightarrow \frac{x}{y} = \frac{\cos \alpha - \cos \beta}{\sin \alpha - \sin \beta} = \frac{-2 \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)}{2 \sin\left(\frac{\alpha-\beta}{2}\right) \cos\left(\frac{\alpha+\beta}{2}\right)}$$

$$= -\tan\left(\frac{\alpha+\beta}{2}\right)$$

or $x + \tan\left(\frac{\alpha+\beta}{2}\right)y = 0$

5. As we know that the centroid of triangle ABC and that of the triangle formed by joining the middle points of the sides of triangle ABC are the same. So, the required centroid is $\left(\frac{4+4-2}{3}, \frac{5-3+3}{3}\right) \equiv \left(2, \frac{5}{3}\right)$

6. Here $AB = BC = CA = 2$.

So, it is an equilateral triangle and the incentre coincides with centroid.

Therefore, centroid

$$\left(\frac{0+1+2}{3}, \frac{0+0+\sqrt{3}}{3}\right) \equiv \left(1, \frac{1}{\sqrt{3}}\right)$$

7. Let the given triangle be ABC .

Centroid is $G(1, 3)$ and two vertices of triangle are $B(4, -8)$ and $C(-9, 7)$.

$$\text{Area of triangle } GBC = \frac{1}{2} \begin{vmatrix} 1 & 4 \\ 4 & -8 \\ -9 & 7 \end{vmatrix}$$

$$= \frac{1}{2} |[-8 - 16 + 28 - 72 - 36 - 7]|$$

$$= 55.5 \text{ sq. units}$$

Hence, area of triangle $ABC = 3 \times 55.5 = 166.5$ sq. units

8. Since the circumcenter is at the origin, the orthocenter is

$$(x_1 + x_2 + x_3, x_1 \tan \theta_1 + x_2 \tan \theta_2 + x_3 \tan \theta_3)$$

$$\therefore a = x_1 + x_2 + x_3$$

$$\text{and } b = x_1 \tan \theta_1 + x_2 \tan \theta_2 + x_3 \tan \theta_3$$

$$\text{Also } x_1^2 + x_1^2 \tan^2 \theta_1^2 = x_2^2 + x_2^2 \tan^2 \theta_2^2 = x_3^2 + x_3^2 \tan^2 \theta_3^2$$

$$\text{or } x_1 \sec \theta_1 = x_2 \sec \theta_2 = x_3 \sec \theta_3 = \lambda \text{ (say)}$$

$$\text{Now, } \frac{a}{b} = \frac{x_1 + x_2 + x_3}{x_1 \tan \theta_1 + x_2 \tan \theta_2 + x_3 \tan \theta_3}$$

$$= \frac{\lambda \cos \theta_1 + \lambda \cos \theta_2 + \lambda \cos \theta_3}{\lambda \cos \theta_1 \tan \theta_1 + \lambda \cos \theta_2 \tan \theta_2 + \lambda \cos \theta_3 \tan \theta_3}$$

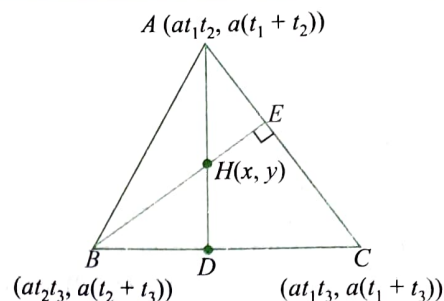
$$= \frac{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}$$

9. $(x_1 + 2)^2 + (y_1 - 3)^2 = (x_2 + 2)^2 + (y_2 - 3)^2 = (x_3 + 2)^2 + (y_3 - 3)^2$
 Therefore, the circumcenter of the triangle formed by points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $(-2, 3)$. But the triangle is equilateral; so, the centroid is $(-2, 3)$. Therefore,

$$\frac{x_1 + x_2 + x_3}{3} = -2, \quad \frac{y_1 + y_2 + y_3}{3} = 3$$

$$\text{or } \frac{x_1 + x_2 + x_3}{y_1 + y_2 + y_3} = -\frac{2}{3}$$

10.



We know that orthocenter of Δ is point of concurrency of altitudes. Let orthocenter be $H(x, y)$.

$$\text{Now, slope of } BC = \frac{a(t_1 + t_3) - a(t_2 + t_3)}{at_1t_3 - at_2t_3}$$

$$= \frac{a(t_1 + t_3 - t_2 - t_3)}{at_3(t_1 - t_2)} = \frac{1}{t_3}$$

$$\therefore \text{Slope of } AD = -t_3$$

$$\text{Also, slope of } AD = \text{Slope of } AH = \frac{y - a(t_1 + t_2)}{x - at_1t_2} = -t_3$$

$$\therefore y - a(t_1 + t_2) = -t_3(x - at_1t_2)$$

$$\text{or } xt_3 + y = at_1t_2t_3 + a(t_1 + t_2) \quad (1)$$

Similarly, by symmetry from $AC \perp BE$, we get

$$xt_1 + y = at_1t_2t_3 + a(t_2 + t_3) \quad (2)$$

Solving (1) and (2), we get

$$x = -a, y = a(t_1 + t_2 + t_3) + at_1t_2t_3$$

$$\therefore \text{Orthocenter is } H(-a, a(t_1 + t_2 + t_3) + at_1t_2t_3)$$

Exercise 1.4

1. The slope joining points $A(x, 2x)$ and $B(3, 5)$ is $(2x - 5)/(x - 3)$. If AB makes an angle θ with the positive direction of the x -axis, then

$$\tan \theta = \frac{2x - 5}{x - 3}$$

Since θ is obtuse, $\tan \theta < 0$

$$\text{or } \frac{2x - 5}{x - 3} < 0$$

$$\text{or } x \in \left(\frac{5}{2}, 3\right)$$

2. The slope of line passing through $(4, 3)$ and $(2, k)$

$$m_1 = \frac{k - 3}{2 - 4} = \frac{3 - k}{2}$$

The slope of the given line $y = 2x + 3$,

$$m_2 = 2$$

The lines are parallel. Therefore,

$$\frac{3 - k}{2} = 2$$

$$\text{or } k = -1$$

3. Let the coordinates of point A be (h, k) .

Midpoint of BC is $D(35/2, 39/2)$.

Slope of AD is

$$\frac{\frac{39}{2} - k}{\frac{35}{2} - h} = -5 \quad (\text{given})$$

$$\Rightarrow 39 - 2k = -5(35 - 3h)$$

$$\Rightarrow 39 - 2k = -175 + 10h$$

$$\Rightarrow 5h + k = 107$$

Also, area of ΔABC is 70 sq. units.

(1)

$$\therefore \begin{vmatrix} h & k & 1 \\ 12 & 19 & 1 \\ 23 & 20 & 1 \end{vmatrix} = \pm 140$$

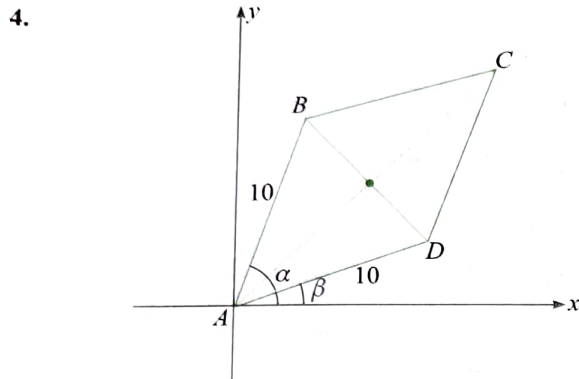
$$\Rightarrow 11k - h = 337 \quad (2)$$

$$\text{or } 11k - h = 57 \quad (3)$$

Solving (1) and (2), we get $h = 15, k = 32$

Solving (1) and (3), we get $h = 20, k = 7$

So, possible coordinates of A are $(15, 32)$ or $(20, 7)$.



In the figure, $\tan \alpha = \frac{4}{3}$.

$\therefore$ Coordinates of point B

$$\equiv (10 \cos \alpha, 10 \sin \alpha)$$

$$\equiv \left(10 \times \frac{3}{5}, 10 \times \frac{4}{5} \right) \equiv (6, 8)$$

Also, $\tan \beta = \frac{3}{4}$.

$\therefore$ Coordinates of point D

$$\equiv (10 \cos \beta, 10 \sin \beta)$$

$$\equiv \left(10 \times \frac{4}{5}, 10 \times \frac{3}{5} \right) \equiv (8, 6)$$

Diagonals of rhombus bisect.

So, midpoint of BD is $(7, 7)$, which is midpoint of AC .

Therefore, coordinates of C are $(14, 14)$.

5. The slope of the line joining $A(2, 1)$ and $B(3, 2)$ is

$$\frac{2-1}{3-2} = 1$$

The slope of the line $(a^2)x + (a+2)y + 2 = 0$ is

$$-\frac{a^2}{a+2}$$

The lines are perpendicular. Therefore,

$$\left(-\frac{a^2}{a+2} \right)(1) = -1$$

$$\text{or } a^2 = a + 2$$

$$\text{or } a^2 - a - 2 = 0$$

$$\text{or } (a-2)(a+1) = 0$$

$$\text{or } a = 2, -1$$

6. $x + 2y - 7 = 0$

$$\text{Slope of line (i)} = m_1 = -\frac{1}{2}$$

$$\text{Slope of line } PQ = m_2 = \frac{2 - (-2)}{3 - 1} = 2$$

where $P \equiv (1, -2)$ and $Q \equiv (3, 2)$.

Since $m_1 \cdot m_2 = -1$ the angle between line (1) and line PQ is $\pi/2$.

7. Let the coordinates of vertex A be (x, y) .

$$AH \perp BC$$

$$\text{or } (\text{Slope of } AH) \times (\text{Slope of } BC) = -1$$

$$\text{or } \frac{y+1}{x-3} \times \frac{-2-0}{1-(-2)} = -1$$

$$\text{or } -2y - 2 = 9 - 3x$$

$$\text{or } 3x - 2y = 11 \quad (1)$$

$$\text{Also, } BH \perp AC$$

$$\text{or } (\text{Slope of } BH) \times (\text{Slope of } AC) = -1$$

$$\text{or } \frac{-2-(-1)}{1-3} \times \frac{y-0}{x-(-2)} = -1$$

$$\text{or } -y = 2x + 4$$

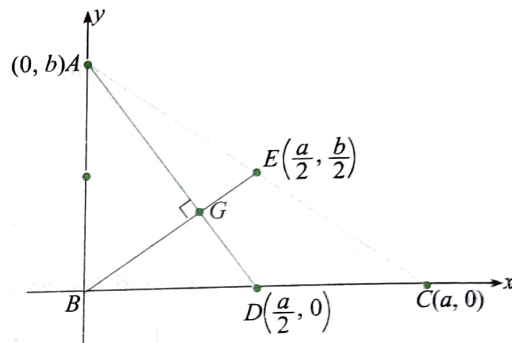
$$\text{or } 2x + y = -4 \quad (2)$$

Solving (1) and (2), we get

$$x = \frac{3}{7}, y = -\frac{34}{7}$$

Hence, the orthocenter is $(3/7, -34/7)$.

8.



From the figure,

$$\text{Slope of } BE, m_{BE} = \frac{b}{a}$$

$$\text{Slope of } AD, m_{AD} = -\frac{2b}{a}$$

Given that AD and BE are perpendicular.

$$\therefore m_{BE} \times m_{AD} = -\frac{2b^2}{a^2} = -1$$

$$\Rightarrow a^2 = 2b^2$$

Exercise 1.5

1. (i) $(2, \pi) \equiv (r, \theta)$

$$\therefore x = r \cos \theta = 2 \cos \pi = -2$$

$$\text{and } y = r \sin \theta = 2 \sin \pi = 0$$

Therefore, the rectangular coordinates are $(x, y) \equiv (-2, 0)$.

(ii) $\left(\sqrt{3}, \frac{\pi}{6} \right) \equiv (r, \theta)$

$$\therefore x = \sqrt{3} \cos \frac{\pi}{6} = \sqrt{3} \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{2}$$

$$\text{and } y = \sqrt{3} \sin \frac{\pi}{6} = \sqrt{3} \left(\frac{1}{2} \right) = \frac{\sqrt{3}}{2}$$

The rectangular coordinates are $(x, y) \equiv (3/2, \sqrt{3}/2)$.

2. (i) Given Cartesian coordinates are $(1, -1)$.

For r to be positive,

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ and } \tan \theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can

$$\text{choose } \theta = -\frac{\pi}{4}$$

So, polar coordinates are $(\sqrt{2}, -\pi/4)$.

Thus, the possible polar coordinates are $(\sqrt{2}, -\pi/4)$; another possibility is $(\sqrt{2}, 7\pi/4)$.

- (ii) Given Cartesian coordinates are $(-3, -4)$.
For r to be positive,

$$r = \sqrt{(-3)^2 + (-4)^2} = 5$$

$$\text{and } \tan \theta = \frac{-4}{-3} = \frac{4}{3} \text{ or } \theta = \tan^{-1} \frac{4}{3}$$

Since the point $(-4, -3)$ lies in the third quadrant, we can choose $\theta = \pi + \tan^{-1} \frac{4}{3}$ or $\theta = -\pi + \tan^{-1} \frac{4}{3}$.

Thus, the possible polar coordinates are $\left(5, \pi + \tan^{-1} \frac{4}{3}\right)$; another possibility is $\left(5, -\pi + \tan^{-1} \frac{4}{3}\right)$.

$$\begin{aligned} 3. \quad & 2x^2 + 3y^2 = 6 \\ \text{or} \quad & 2(r \cos \theta)^2 + 3(r \sin \theta)^2 = 6 \\ \text{or} \quad & 2r^2 \cos^2 \theta + 3r^2 \sin^2 \theta = 6 \\ \text{or} \quad & 2r^2 + r^2 \sin^2 \theta = 6 \\ \text{or} \quad & r^2 (2 + \sin^2 \theta) = 6 \end{aligned}$$

$$4. \text{ Given } r = 4 \tan \theta \sec \theta$$

$$\text{or } r = \frac{4 \sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta}$$

$$\text{or } r \cos^2 \theta = 4 \sin \theta$$

$$\text{or } r^2 \cos^2 \theta = 4 r \sin \theta$$

$$\text{or } x^2 = 4y$$

5. Let the polar coordinates of any point on the curve be $P(r, \theta)$.
Then its Cartesian coordinates are $P(r \cos \theta, r \sin \theta)$.

P lies on the line. Therefore,

$$3r \cos \theta + 4r \sin \theta - 10 = 0$$

$$\text{or } r = \frac{10}{3 \cos \theta + 4 \sin \theta}$$

$$\text{Now, } (3 \cos \theta + 4 \sin \theta)_{\max} = 5$$

$$\therefore r_{\min} = \frac{10}{5} = 2$$

Exercise 1.6

1. Let the point be (h, k) . Therefore,

$$(h - a)^2 + (k - 0)^2 = h^2$$

$$\text{or } h^2 + a^2 - 2ah + k^2 = h^2$$

$$\text{Hence, the locus is } y^2 - 2ax + a^2 = 0.$$

2. Let the point be (x, y) . Then,

$$(x - a)^2 + y^2 - (x + a)^2 - y^2 = 2k^2$$

$$\text{or } -4ax - 2k^2 = 0$$

$$\text{or } 2ax + k^2 = 0$$

This is the required equation to the locus of point P .

3. Let C be (α, β) .

The centroid is

$$\left(\frac{2 - 2 + \alpha}{3}, \frac{-3 + 1 + \beta}{3}\right), \text{ i.e., } \left(\frac{\alpha}{3}, \frac{\beta - 2}{3}\right)$$

This lies on $2x + 3y = 1$. Therefore, we get

$$2\left(\frac{\alpha}{3}\right) + 3\left(\frac{\beta - 2}{3}\right) = 1$$

$$\text{or } 2\alpha + 3\beta = 9$$

Hence, the locus of (α, β) is $2x + 3y = 9$.

4. Let Q be the point (X, Y) and P be the point (x, y) ; the coordinates of Q satisfy the equation $2x + 3y + 4 = 0$, so that $2X + 3Y + 4 = 0$.

Applying the section formula for OQ , O being $(0, 0)$, we get

$$x = \frac{0 + 3X}{1 + 3}, y = \frac{0 + 3Y}{1 + 3}$$

from which we get

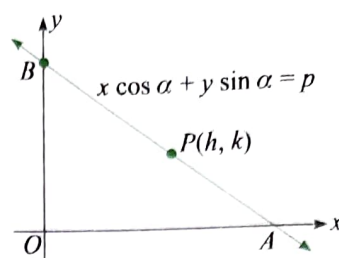
$$X = \frac{4}{3}x, Y = \frac{4}{3}y$$

Substituting these values, the locus of P is

$$\frac{8}{3}x + 4y + 4 = 0$$

$$\text{or } 2x + 3y + 3 = 0$$

5.



The equation of the variable line is

$$x \cos \alpha + y \sin \alpha = p \quad (1)$$

Here p is a constant and α is the parameter (variable).

Let the line in (1) cuts x - and y -axes at A and B , respectively.

Putting $y = 0$ in (1), we get $A \equiv (p \sec \alpha, 0)$.

Putting $x = 0$ in (1), we get $B \equiv (0, p \csc \alpha)$.

AB is the portion of (1) intercepted between the axes.

Let $P(h, k)$ be the midpoint of AB . We have to find the locus of point $P(h, k)$. For this, we will have to eliminate α and find a relation in h and k . Therefore,

$$h = \frac{p \sec \alpha + 0}{2} = \frac{p}{2} \sec \alpha \quad (2)$$

$$\text{and } k = \frac{0 + p \csc \alpha}{2} = \frac{p}{2} \csc \alpha \quad (3)$$

From (2), we get

$$\cos \alpha = \frac{p}{2h} \quad (4)$$

From (3), we get

$$\sin \alpha = \frac{p}{2k} \quad (5)$$

Squaring and adding (4) and (5), we get

$$\cos^2 \alpha + \sin^2 \alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2}$$

$$\text{or } \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2}$$

Hence, the locus of point $P(h, k)$ is

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$$

6. Let (h, k) be the point of intersection of $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$. Then,

$$h \cos \alpha + k \sin \alpha = a \quad (1)$$

$$h \sin \alpha - k \cos \alpha = b \quad (2)$$

Squaring and adding (1) and (2), we get

$$(h \cos \alpha + k \sin \alpha)^2 + (h \sin \alpha - k \cos \alpha)^2 = a^2 + b^2$$

$$\text{or } h^2 + k^2 = a^2 + b^2$$

Hence, the locus of (h, k) is

$$x^2 + y^2 = a^2 + b^2$$

7. Let (x, y) be the required point. Therefore,

$$\begin{vmatrix} x & y \\ 1 & 5 \\ 2 & -7 \\ x & y \end{vmatrix} = \pm 21$$

$$\text{or } 5x - y - 7 - 15 + 3y + 7x = \pm 21$$

$$\text{i.e., } 12x + 2y = 64 \text{ or } 12x + 2y = -20$$

$$\text{i.e., } 6x + y = 32 \text{ or } 6x + y = -10$$

8. Since triangle OAB is right-angled, its circumcenter is the midpoint of hypotenuse AB .
 So, let the midpoint of AB be $Q(h, k)$.
 Then the coordinates of A and B are $(2h, 0)$ and $(0, 2k)$, respectively.
 Now, points A, B , and P are collinear. Therefore,

$$\begin{vmatrix} 2h & 0 \\ 2 & 1 \\ 0 & 2k \\ 2h & 0 \end{vmatrix} = 0$$

$$\text{or } 2h + 4k - 4hk = 0$$

$$\text{or } x + 2y - 2xy = 0$$

9. Let the point C be (h, k) .

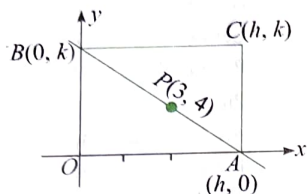
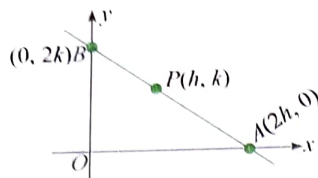
Then the coordinates of A and B are $(h, 0)$ and $(0, k)$, respectively.
 Since points A, P , and B are collinear, we have

$$\begin{vmatrix} h & 0 \\ 3 & 4 \\ 0 & k \\ h & 0 \end{vmatrix} = 0$$

$$\text{or } 4h + 3k - hk = 0$$

$$\text{or } 4x + 3y - xy = 0$$

Therefore, the required locus is $\frac{3}{x} + \frac{4}{y} = 1$.



$$= c^2 + 16 + \frac{116 + c^2 - 20c}{4}$$

$$= \frac{5}{4}c^2 - 5c + 45 = \frac{5}{4}(c^2 - 4c + 36)$$

Therefore, T is minimum at $c = 2$.

5. (1) We have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\text{or } \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

Hence, the area of triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is the same as the area of triangle with vertices $(a_1, b_1), (a_2, b_2), (a_3, b_3)$. Hence, the two triangles are equal in area.

6. (3) Let the coordinates of vertices O, P, Q, R be $(0, 0), (a, 0), (a, a), (0, a)$, respectively. Then, we get the coordinates of M as $(a, a/2)$ and those of N as $(a/2, a)$.

Therefore, the area of $\triangle OMN$ is

$$\frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a & a/2 & 1 \\ a/2 & a & 1 \end{vmatrix} = \frac{3a^2}{8}$$

The area of the square is a^2 . Hence, the required ratio is $8:3$.

7. (1) Line AB will be the farthest from the origin if OP is right angled to the line drawn.

$$OP = \sqrt{10}$$

$$\text{Also, } \tan \theta = \frac{1}{3}$$

$$OA = OP \sec \theta$$

$$= \sqrt{10} \times \frac{\sqrt{10}}{3} = \frac{10}{3}$$

$$OB = OP \csc \theta = \sqrt{10} \times \frac{\sqrt{10}}{1} = 10$$

$$\therefore \text{Area of } \triangle OAB = \frac{1}{2} \left(\frac{10}{3} \right) (10) = \frac{50}{3}$$

8. (3) We know that $PA + PB \geq AB$ (by triangle inequality). So, $PA + PB$ is the minimum if $PA + PB = AB$, i.e., A, P, B are collinear. Therefore,

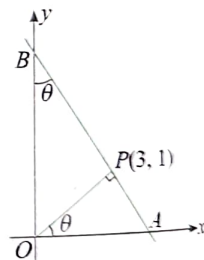
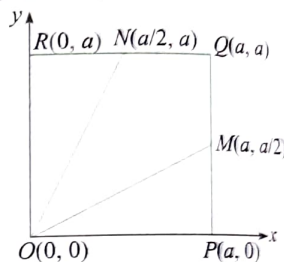
$$\begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 1 \\ 2k-1 & 2k+1 & 1 \end{vmatrix} = 0$$

$$\text{or } 3(2 - 2k - 1) + 4(1 - 2k + 1) + 1(2k + 1 - 4k + 2) = 0$$

$$\text{or } 3 - 6k + 8 - 8k + 3 - 2k = 0$$

$$\text{or } 14 - 16k = 0$$

$$\therefore k = \frac{7}{8}$$



Exercises

Single Correct Answer Type

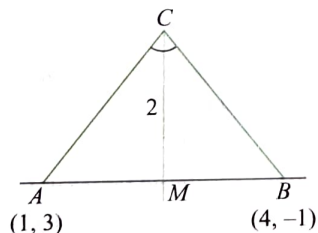
1. (3) Let the vertex of triangle be $A(x, y)$.
 Then the vertex $A(x, y)$ is equidistant from B and C because ABC is an isosceles triangle. Therefore,
 $(x - 1)^2 + (y - 3)^2 = (x + 2)^2 + (y - 7)^2$
 or $6x - 8y + 43 = 0$

Thus, any point lying on this line can be the vertex A except the midpoint $(-1/2, 5)$ of BC . Hence, vertex A is $(5/6, 6)$.

2. (1) Length of base $AB = 5$
 $\therefore$ Length of altitude $CM = 2$
 ($\because$ Area is 5 sq. units)

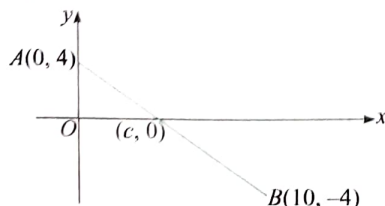
$$\text{Max. } \angle ACB = 2 \tan^{-1} \frac{5}{4}$$

$$\therefore \angle ACB \in \left(0, 2 \tan^{-1} \frac{5}{4} \right]$$



3. (4) For the equilateral triangle, one vertex must be an irrational point.

4. (2) Let the point on the x -axis be $(c, 0)$.
 Sum of the squares of travel times is



$$T = \left(\frac{\sqrt{c^2 + 16}}{1} \right)^2 + \left(\frac{\sqrt{(10 - c)^2 + 16}}{2} \right)^2$$

$$9. (4) r = \sqrt{x^2 + y^2} = \sqrt{9 + 3} = 2\sqrt{3}$$

$$\alpha = \tan^{-1} \frac{|y|}{|x|}$$

$$\alpha = \tan^{-1} \left(\frac{\sqrt{3}}{3} \right)$$

$$\alpha = \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \frac{\pi}{6}$$

Point $(-3, \sqrt{3})$ lies in the second quadrant.

$$\therefore \theta = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

So, polar coordinates are $\left(2\sqrt{3}, \frac{5\pi}{6} \right)$.

10. (2) $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)) \equiv (x_1(1 - t) + tx_2, y_1(1 - t) + ty_2)$ is the point which divides the join of (x_1, y_1) and (x_2, y_2) in the ratio $t : (1 - t)$, which is positive if $0 < t < 1$.

11. (2) The midpoint of PQ is the midpoint of AB which is $R(1/2, 3)$. Therefore,

$$OR = \sqrt{\frac{1}{4} + 9} = \frac{\sqrt{37}}{2}$$

12. (2) $AB = \sqrt{2}$, $AC = 2$, $BC = \sqrt{2}$

Using Apollonius' Theorem, we get

$$2(AD^2 + BD^2) = AC^2 + AB^2$$

$$\therefore 2(AD^2 + 1/2) = 4 + 2$$

$$\therefore AD^2 + 1/2 = 3$$

$$\therefore AD^2 = 5/2$$

$$\therefore AD = \sqrt{\frac{5}{2}}$$

13. (1) Given are centroid $G\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and one vertex $A(2, 2)$.

$\therefore$ Length of side of triangle

$$= \sqrt{3} GA$$

$$= \sqrt{3} \sqrt{\left(-\frac{2}{\sqrt{3}} - 2\right)^2 + \left(\frac{2}{\sqrt{3}} - 2\right)^2}$$

$$= 2\sqrt{(1 + \sqrt{3})^2 + (1 - \sqrt{3})^2} = 2\sqrt{8} = 4\sqrt{2}$$

14. (4) Since $ABCD$ is a rectangle,

$$(AC)^2 = (AB)^2 + (BC)^2$$

$$= (AB)^2 \left[1 + \frac{9}{16} \right] = \frac{25}{16} (AB)^2$$

$$\text{and } PA = \frac{AC}{2} = \frac{5AB}{8}$$

$$= \frac{5}{8} \sqrt{(-1 - 3)^2 + (2 - 7)^2} = 5 \frac{\sqrt{41}}{8}$$

15. (4) We have $A(2, -3)$, $B(6, 5)$, $C(-2, 1)$

Area of rhombus $ABCD$

$$= 2 \times \text{Area of } \triangle ABC$$

$$= 2 \times 24 = 48$$

16. (4) Let $B \equiv (\alpha, \beta)$, where α, β are rational.

$$\text{Given } HA^2 = HB^2$$

$$\Rightarrow (3 - \sqrt{2})^2 + (5 - \sqrt{5})^2 = (\alpha - \sqrt{2})^2 + (\beta - \sqrt{5})^2$$

$$\Rightarrow 41 - 6\sqrt{2} - 10\sqrt{5} = \alpha^2 + \beta^2 + 7 - 2\sqrt{2}\alpha - 2\sqrt{5}\beta$$

$$\Rightarrow \alpha^2 + \beta^2 - 34 + 2(3 - \alpha)\sqrt{2} + 2(5 - \beta)\sqrt{5} = 0$$

Since α and β are rational,

$$3 - \alpha = 0 \Rightarrow \alpha = 3$$

$$\text{and } 5 - \beta = 0 \Rightarrow \beta = 5$$

$$\therefore AB = 0$$

17. (3) Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points at equal distance a from $P(0, \sqrt{3})$. Then

$$x_1^2 + (y_1 - \sqrt{3})^2 = a^2 \quad (1)$$

$$\text{and } x_2^2 + (y_2 - \sqrt{3})^2 = a^2 \quad (2)$$

Subtracting (2) from (1), we get

$$x_1^2 - x_2^2 + y_1^2 - y_2^2 - 2\sqrt{3}(y_1 - y_2) = 0$$

Equating irrational and rational parts to zero, we get

$$y_1 = y_2$$

$$\text{and } x_1^2 - x_2^2 + y_1^2 - y_2^2 = 0$$

Thus, $y_1 = y_2 = a$ (say)

and $x_1^2 = x_2^2 = b^2$ (say)

Hence, at most two such points are possible whose coordinates are (b, a) and $(-b, a)$.

In a special case, when $b = 0$, only one such point is possible.

18. (4) We have $a = 5$, $b = 4$, $c = 3$

Incentre I divides AD in the ratio $b + c : a$.

Therefore, I divides AD in the ratio $7 : 5$.

Hence, coordinates of I are $(1, 1)$.

19. (3) $AB = AC = BC = 1$

So, triangle is an equilateral.

Thus, incentre coincide with centroid.

Centre of the circle touching all the three sides of triangle is incentre.

Therefore, incentre is

$$\left(\frac{0 + 1 + \frac{1}{2}}{3}, \frac{0 + 0 + \frac{\sqrt{3}}{2}}{3} \right) \equiv \left(\frac{1}{2}, \frac{1}{2\sqrt{3}} \right)$$

20. (4) Statement 2 is obviously correct.

For Statement 1, let the circumcentre be at $(0, 0)$ and the vertices of the triangle be (x_1, y_1) , (x_2, y_2) and (x_3, y_3) . Then

centroid is $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$, and orthocentre of

the triangles becomes $(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$.

This implies that if the centroid is rational then orthocentre is also rational but $(x_1 + x_2 + x_3)$ can be rational even if x_1, x_2, x_3 are not all rational.

For example, $A(1, 0)$, $B(-1/2, \sqrt{3}/2)$ and $C(-1/2, -\sqrt{3}/2)$, where G, H and C are at $(0, 0)$ i.e., rational points

21. (4) $P \equiv (-\sin(\beta - \alpha), -\cos \beta) \equiv (x_1, y_1)$

$$Q \equiv (\cos(\beta - \alpha), \sin \beta) \equiv (x_2, y_2)$$

$$R \equiv (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$$

$$\text{or } R \equiv (x_2 \cos \theta + x_1 \sin \theta, y_2 \cos \theta + y_1 \sin \theta)$$

$$\text{If } T \equiv \left(\frac{x_2 \cos \theta + x_1 \sin \theta}{\cos \theta + \sin \theta}, \frac{y_2 \cos \theta + y_1 \sin \theta}{\cos \theta + \sin \theta} \right), \text{ for some } \theta'$$

then P, Q , and T are collinear.

$\therefore P, Q, R$ are collinear if $\cos \theta + \sin \theta = 1$, which is not possible as $0 < \theta < \pi/4$.

Hence, P, Q, R are non-collinear.

22. (4) Given $O(0, 0)$ is the orthocenter. Let $A(h, k)$ be the third vertex, and $B(-2, 3)$ and $C(5, -1)$ the other two vertices. Then the slope of the line through A and O is k/h , while the line through B and C has the slope $-4/7$. By the property of the orthocenter, these two lines must be perpendicular. So, we have

$$\left(\frac{k}{h}\right)\left(-\frac{4}{7}\right) = -1 \text{ or } \frac{k}{h} = \frac{7}{4} \quad (i)$$

$$\text{Also, } \frac{5-2+h}{3} + \frac{-1+3+k}{3} = 7$$

$$\text{or } h+k=16 \quad (ii)$$

which is not satisfied by the points given in (1), (2), or (3).

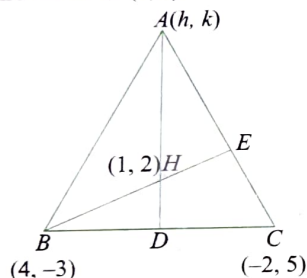
23. (2) p, q, r are the roots of equation $y^3 - 3y^2 + 6y + 1 = 0$. So, $p+q+r=3$, $pq+qr+rp=6$, and $pqr=-1$. Now, the centroid of the triangle is

$$\left(\frac{pq+qr+rp}{3}, \frac{\frac{1}{p}+\frac{1}{q}+\frac{1}{r}}{3}\right)$$

$$\text{i.e., } \left(\frac{pq+qr+rp}{3}, \frac{p+q+r}{3pqr}\right) \equiv \left(\frac{6}{3}, \frac{3}{-3}\right) \text{ or } (2, -1)$$

24. (3) The circumcenter of the triangle is $(0, 0)$ as vertices are at equal distance from $(0, 0)$. So, the orthocenter will be (sum of x -coordinates, sum of y -coordinates).

25. (2) Let the third vertex be (h, k) .



Now, the slope of AD is $(k-2)/(h-1)$, the slope of BC is $(5+3)/(-2-4) = -4/3$, the slope of BE is $(-3-2)/(4-1) = -5/3$, and the slope of AC is $(k-5)/(h+2)$. Since $AD \perp BC$, we have

$$\frac{k-2}{h-1} \times \frac{-4}{3} = -1$$

$$\text{or } 3h-4k+5=0 \quad (1)$$

Again, since $BE \perp AC$, we have

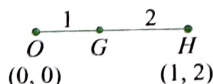
$$-\frac{5}{3} \times \frac{k-5}{h+2} = -1$$

$$\text{or } 3h-5k+31=0 \quad (2)$$

On solving (1) and (2), we get $h=33, k=26$. Hence, the third vertex is $(33, 26)$.

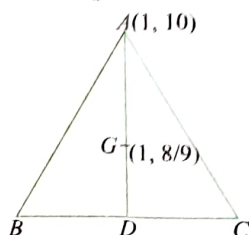
$$26. (2) G \equiv \left(\frac{1 \times 1 + 2 \times 0}{3}, \frac{2 \times 1 + 2 \times 0}{3}\right)$$

$$\equiv \left(\frac{1}{3}, \frac{2}{3}\right)$$

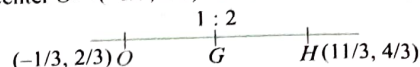


27. (4) The orthocenter of triangle BCH is the vertex $A(-1, 0)$.

28. (1)



Circumcenter $O \equiv (-1/3, 2/3)$ and orthocenter $H \equiv (11/3, 4/3)$.



Therefore, the coordinates of G are $(1, 8/9)$. Now, the point A is $(1, 10)$ as G is $(1, 8/9)$.

Also, $AD : DG = 3 : 1$

$$\therefore D \equiv \left(\frac{3-1}{3-1}, \frac{8/3-10}{3-1}\right) \equiv (1, -11/3)$$

$$\therefore D = \frac{3-1}{2} = 1, D_y = \frac{(8/3)-10}{2} = -\frac{11}{3}$$

Hence, the coordinates of the midpoint of BC are $(1, -11/3)$.

29. (2) $(2, 0)$ is the midpoint of $B(0, 0)$ and C . Then C has coordinates $(4, 0)$. Also, A has coordinates $(0+2 \cos \pi/3, 0+2 \sin \pi/3) \equiv (1, \sqrt{3})$. Then the centroid is $(5/3, 1/\sqrt{3})$.

30. (4) The given equation is

$$(a-b)(x^2+y^2)-2abx=0 \quad (1)$$

The origin is shifted to $(ab/(a-b), 0)$. Any point (x, y) on the curve (1) must be replaced with a new point (X, Y) with reference to the new axes, such that

$$x = X + \frac{ab}{a-b} \text{ and } y = Y + 0$$

Substituting these in (i), we get

$$(a-b)\left[\left(X+\frac{ab}{a-b}\right)^2+Y^2\right]-2ab\left[X+\frac{ab}{a-b}\right]=0$$

$$\text{or } (a-b)\left[X^2+\frac{a^2b^2}{(a-b)^2}+Y^2+\frac{2abX}{a-b}\right]-2abX-\frac{2a^2b^2}{a-b}=0$$

$$\text{or } (a-b)(X^2+Y^2)=\frac{a^2b^2}{a-b}$$

$$\text{or } (a-b)^2(X^2+Y^2)=a^2b^2$$

31. (3) The point of coincidence on the y -axis is $Q(0, \lambda)$.

The image of $P(2, 3)$ on the y -axis is $P_1(-2, 3)$.

P_1, Q , and R are collinear.

Therefore,

$$\text{Slope of } P_1Q = \text{Slope of } P_1R$$

$$\text{or } \frac{\lambda-3}{0-(-2)} = \frac{10-3}{5-(-2)}$$

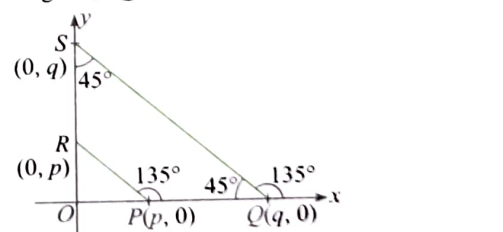
$$\text{or } \lambda-3=2$$

$$\text{or } \lambda=5$$

Therefore, the point Q is $(0, 5)$.

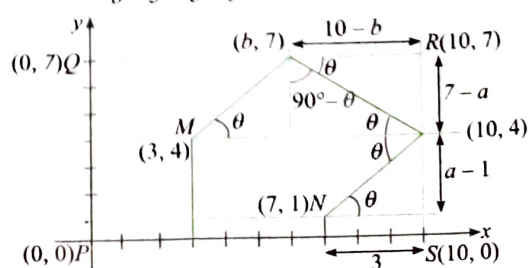
32. (3) The diagram is self-explanatory.

From the diagram, $PQRS$ is a cyclic quadrilateral.



$$33. (1) \tan \theta = \frac{a-1}{3} = \frac{7-a}{10-b}$$

$$\text{Also, } \tan \theta = \frac{7-4}{b-3} = \frac{3}{b-3}$$



$$\text{Hence, } \frac{3}{b-3} = \frac{a-1}{3} = \frac{7-a}{10-b}$$

From the first two relations, we have

$$9 = ab - b - 3a + 3$$

$$\text{or } 3a + 6 = ab - b$$

From the last two relations, we have

$$10a - ab - 10 + b = 21 - 3a$$

$$\text{or } 13a - ab + b = 31$$

$$\text{or } ab - b = 13a - 31$$

Hence, from (1) and (2),

$$3a + 6 = 13a - 31$$

$$\text{or } 10a = 37$$

$$\text{or } a = 3.7$$

34. (4) Since EF divides the square in two equal parts, it must pass through the center of the square and center is midpoint of EF . So, center of the square is $(3, 4)$.

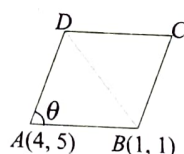
Therefore, coordinates of C are $(-1, 5)$.

35. (2) From the figure,

$$\text{Area of rhombus} = 2 \times (\text{Area of } \triangle ABD)$$

$$= 2 \times \frac{1}{2} \times 5 \times 5 \sin \theta$$

$$= 25 \sin \theta$$



Hence, the maximum area is 25 (when $\sin \theta = 1$).

36. (4) Clearly, A will remain as $(0, 0)$; f_1 will make B as $(0, 4)$, f_2 will make it $(12, 4)$, and f_3 will make it $(4, 8)$; f_1 will make C as $(2, 4)$, f_2 will make it $(14, 4)$, and f_3 will make it $(5, 9)$. Finally, f_1 will make D as $(2, 0)$, f_2 will make it $(2, 0)$, and f_3 will make it $(1, 1)$. So, we finally get $A \equiv (0, 0)$, $B \equiv (4, 8)$, $C \equiv (5, 9)$, and $D \equiv (1, 1)$. Hence,

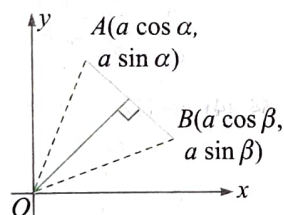
$$m_{AB} = \frac{8}{4}, m_{BC} = \frac{9-8}{5-4} = 1, m_{CD} = \frac{9-1}{5-1} = \frac{8}{4},$$

$$m_{AD} = 1, m_{AC} = \frac{9}{5}, m_{BD} = \frac{8-1}{4-1} = \frac{7}{3}$$

Hence, the final figure will be a parallelogram.

37. (1) Since $OA = OB = a$, AOB is an isosceles triangle.

The line through the origin bisecting AB will be perpendicular to AB .



38. (2) $OM_r = OA_r + \frac{OA_{r+1} - OA_r}{2}$

$$= \frac{OA_r + OA_{r+1}}{2}$$

$$= \frac{1}{2} \{OA_1 \times k^{r-1} + OA_1 \times k^r\}$$

$$= \frac{OA_1}{2} (1+k) k^{r-1}$$

$$\therefore \sum_{r=1}^{\infty} OM_r = \frac{OA_1}{2} (1+k) \sum_{r=1}^{\infty} k^{r-1}$$

$$= \frac{OA_1}{2} (1+k) \times \frac{1}{1-k}$$

$$= \frac{OA_1}{2} \times \frac{1 + (OA_2/OA_1)}{1 - (OA_2/OA_1)} = \frac{OA_1(OA_1 + OA_2)}{2(OA_1 - OA_2)}$$

39. (2) Slope of line through origin is,

$$m = \frac{21-x}{13} = \frac{x+1}{3}$$

$$\text{or } 63 - 3x = 13x + 13$$

$$\text{or } 16x = 50$$

$$\text{or } x = \frac{25}{8}$$

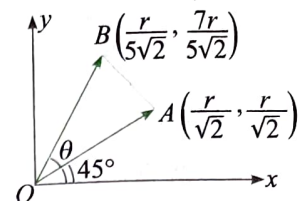
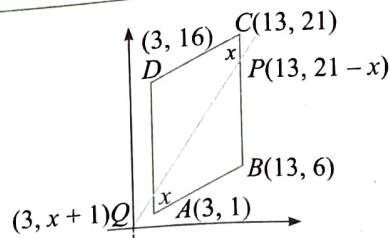
$$\text{Hence, } m = \left(\frac{25}{8} + 1 \right) \times \frac{1}{3} = \frac{33}{24} = \frac{11}{8}$$

40. (4) $\tan \theta = 7$

$$OA = OB = r$$

$$\sin \theta = \frac{7}{5\sqrt{2}}, \cos \theta = \frac{1}{5\sqrt{2}}$$

$$\text{Now, } m_{AB} = -1/2.$$



41. (2) Given that a, b, c are in A.P.

$$\Rightarrow a - b = b - c$$

Now, (a, x) , (b, y) and (c, z) are collinear.

$$\Rightarrow \frac{x-y}{a-b} = \frac{y-z}{b-c}$$

$$\Rightarrow \frac{x-y}{y-z} = \frac{a-b}{b-c} = 1$$

$$\Rightarrow x - y = y - z$$

So, x, y, z are in A.P.

Thus x, y, z are in A.P. and also in G.P.

$$\therefore x = y = z$$

42. (1) Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$, $D \equiv (x_4, y_4)$

Given

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2 - 2x_1x_3 - 2x_2x_4 - 2y_1y_3 - 2y_2y_4 \leq 0$$

$$\text{or } (x_1 - x_3)^2 + (x_2 - x_4)^2 + (y_1 - y_3)^2 + (y_2 - y_4)^2 \leq 0$$

$$\text{or } x_1 = x_3, x_2 = x_4, y_1 = y_3, y_2 = y_4$$

$$\text{or } \frac{x_1 + x_2}{2} = \frac{x_3 + x_4}{2} \text{ and } \frac{y_1 + y_2}{2} = \frac{y_3 + y_4}{2}$$

Hence, AB and CD bisect each other.

Therefore, $ACBD$ is a parallelogram. Also,

$$\begin{aligned} AB^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \\ &= (x_3 - x_4)^2 + (y_3 - y_4)^2 \\ &= CD^2 \end{aligned}$$

Thus, $ACBD$ is a parallelogram and $AB = CD$. Hence, it is a rectangle.

43. (3) Let the coordinates of B be (h, k) .

Draw BL and CM perpendicular to the x - and the y -axis. Therefore,

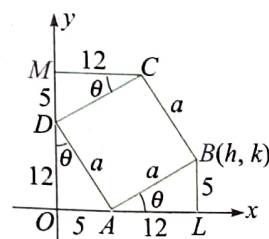
$$a \cos \theta = CM = OD = AL = 12$$

$$\text{and } a \sin \theta = DM = OA = BL = 5$$

$$\therefore k = 5$$

$$\text{and } h = OL = OA + AL = 5 + 12 = 17$$

Hence, point B is $(17, 5)$.



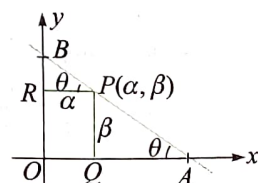
44. (2) $OA = OQ + QA = \alpha + \beta \cot \theta$

$$OB = OR + RB = \beta + \alpha \tan \theta$$

Now, the area of $\triangle OAB$ is

$$\frac{1}{2}(\alpha + \beta \cot \theta)(\beta + \alpha \tan \theta)$$

$$= \frac{1}{2}(2\alpha\beta + \alpha^2 \tan \theta + \beta^2 \cot \theta)$$



$$= \frac{1}{2} \left(2\alpha\beta + (\alpha\sqrt{\tan\theta} - \beta\sqrt{\cot\theta})^2 + 2\alpha\beta \right)$$

$$\geq \frac{1}{2} (2\alpha\beta + 2\alpha\beta)$$

$$= 2\alpha\beta$$

Least value of $S = 2\alpha\beta$

45. (3) Let $(e^t + e^{-t}, e^t - e^{-t}) \equiv (h, k)$

or $h = e^t + e^{-t}$ and $k = e^t - e^{-t}$

Squaring and subtracting, we get

$$h^2 - k^2 = (e^t + e^{-t})^2 - (e^t - e^{-t})^2 = 4$$

Therefore, the locus is $x^2 - y^2 = 4$.

46. (3) Here,

$$x = \frac{a}{2} \left(1 + \frac{1}{t} \right) \quad \dots(i)$$

$$y = \frac{a}{2} \left(1 - \frac{1}{t} \right) \quad \dots(ii)$$

Adding (i) and (ii), we get

$$x + y = a$$

which is the required locus.

47. (1) Let the vertices be $O(0, 0)$, $A(\alpha, 0)$, and $B(\alpha_1, \beta_1)$, where $0 \leq \alpha \leq 1$ and $1 \leq \alpha_1^2 + \beta_1^2 \leq 4$.

So, the area of $\triangle OAB$ is maximum where $\alpha = 1$ and (α_1, β_1) is $(2, 0)$.

In this case, $a = 1$, $b = 2$, and $c = \sqrt{5}$, which satisfies $2 \leq c \leq 3$. Therefore, the maximum area is 1.

48. (4) Distance of all the points from $(0, 0)$ are 5 units. That means the circumcenter of the triangle formed by the given points is $(0, 0)$. If $G \equiv (h, k)$ is the centroid of the triangle, then $3h = 3 + 5(\cos\theta + \sin\theta)$, $3k = 4 + 5(\sin\theta - \cos\theta)$. If $H(\alpha, \beta)$ is the orthocenter, then

$$OG : GH = 1 : 2 \text{ or } \alpha = 3h, \beta = 3k$$

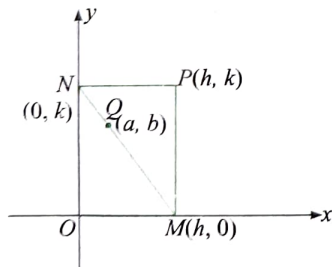
$$\cos\theta + \sin\theta = \frac{\alpha - 3}{5}, \sin\theta - \cos\theta = \frac{\beta - 4}{5}$$

$$\text{or } \sin\theta = \frac{\alpha + \beta - 7}{10}, \cos\theta = \frac{\alpha - \beta + 1}{10}$$

Thus, the locus of (α, β) is

$$(x + y - 7)^2 + (x - y + 1)^2 = 100$$

49. (3)



Let $P(h, k)$

So, points M and N are $(h, 0)$ and $(0, k)$, respectively.

MN passes through the point $Q(a, b)$.

So, M , N and Q are collinear.

$\therefore$ Slope of MN = Slope of QM

$$\therefore \frac{k}{h} = \frac{b - 0}{a - h}$$

$$\therefore ak - hk = -bh$$

So locus is $bx + ay = xy$.

50. (2) Let the point of intersection of given two lines be $P(h, k)$, which lies on both the lines.

$$\therefore k + mh = \sqrt{a^2m^2 + b^2}$$

$$\text{and } mk - h = \sqrt{a^2 + b^2m^2}$$

Squaring and adding, we get

$$(1 + m^2)k^2 + (1 + m^2)h^2 = a^2m^2 + b^2 + a^2 + b^2m^2$$

Therefore, locus is $x^2 + y^2 = a^2 + b^2$.

51. (2) Equation $(x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$ has roots m_1 and m_2 .

$$\therefore m_1m_2 = \frac{y_1^2 + b^2}{x_1^2 - a^2} = -1 \quad (\text{Given})$$

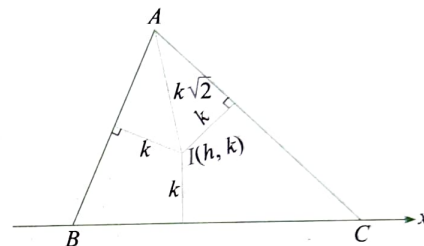
$$\therefore x^2 + y^2 = a^2 - b^2$$

This is the required locus.

52. (2) Let $I(h, k)$ be incentre.

Incentre is equidistant from all sides.

By geometry, $IP = k\sqrt{2}$



$$\therefore \sqrt{(h+1)^2 + (k-4)^2} = k\sqrt{2}$$

$$\therefore h^2 + 1 + 2h + k^2 + 16 - 8k = 2k^2$$

$$\therefore h^2 - k^2 + 2h - 8k + 17 = 0$$

Hence, locus is $x^2 - y^2 + 2x - 8y + 17 = 0$

53. (1) We have

$$3x + 5y = 2007 \text{ or } x + \frac{5y}{3} = 669$$

Clearly, 3 must divide $5y$ and so $y = 3k$, for some $k \in \mathbb{N}$. Thus,

$$x + 5k = 669$$

$$\text{or } 5k \leq 668$$

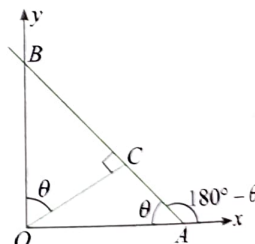
$$\text{or } k \leq \frac{668}{5} \text{ or } k \leq 133$$

54. (4) $\tan(180^\circ - \theta) = \text{Slope of } AB = -3$

$$\therefore \tan\theta = 3$$

$$\therefore \frac{OC}{AC} = \tan\theta, \frac{OC}{BC} = \cot\theta$$

$$\text{or } \frac{BC}{AC} = \frac{\tan\theta}{\cot\theta} = \tan^2\theta = 9$$



55. (4) The point Q is $(-b, -a)$ and the point R is $(-a, -b)$. Therefore, the midpoint of PR is $(0, 0)$.

56. (4) Let (h, k) be the point on the locus. Then by the given conditions,

$$(h - a_1)^2 + (k - b_1)^2 = (h - a_2)^2 + (k - b_2)^2$$

$$\text{or } 2h(a_1 - a_2) + 2k(b_1 - b_2) + a_1^2 + b_1^2 - a_2^2 - b_2^2 = 0$$

$$\text{or } h(a_1 - a_2) + k(b_1 - b_2) + \frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0 \quad (1)$$

Also, since (h, k) lies on the given locus, we have

$$(a_1 - a_2)h + (b_1 - b_2)k + c = 0 \quad (2)$$

Comparing (1) and (2), we get

$$c = \frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2)$$

57. (4) Arranging the lines in descending order of slope, we have

$$m_1 = 5, m_2 = 3, \text{ and } m_3 = -1$$

$$\therefore \tan A = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2}{1 + 15} = \frac{1}{8}$$

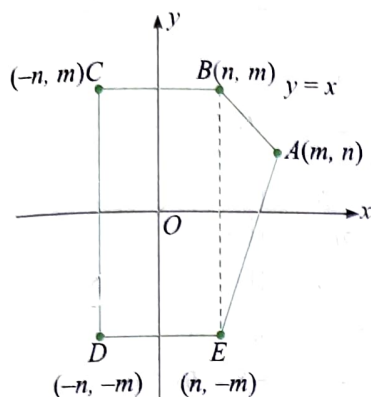
$\left(\because \frac{\theta}{2} \text{ is acute} \right)$

$$\tan B = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{3 + 1}{1 - 3} = -2$$

$$\tan C = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{-1 - 5}{1 - 5} = \frac{3}{2}$$

$$\sum \tan^2 A = \frac{1}{64} + 4 + \frac{9}{4} = \frac{1 + 256 + 144}{64} = \frac{401}{64}$$

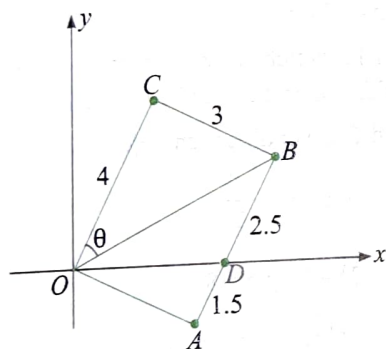
$$\text{or } p + q = 465$$



$$\text{Area of rectangle } BCDE = 4mn$$

$$\text{Area of } \triangle ABC = \frac{2m(m-n)}{2} = m^2 - mn$$

$$\therefore \text{Area of pentagon } ABCDE = 4mn + m^2 - mn = m^2 + 3mn$$



Clearly, $OB = 5$.

$$\text{So, } \frac{OA}{OB} = \frac{3}{5} = \frac{1.5}{2.5} = \frac{AD}{DB}$$

Thus, OD bisects $\angle AOB$.

$$\text{Let } \angle BOC = \theta. \text{ Then } \tan \theta = \frac{3}{4}.$$

$$\therefore \angle BOD = \frac{1}{2} \angle AOB = \frac{1}{2} (90^\circ - \theta)$$

$$\therefore \text{Slope of } OB = \tan \frac{1}{2} (90^\circ - \theta) = \frac{\tan 45^\circ - \tan \frac{\theta}{2}}{1 + \tan 45^\circ \cdot \tan \frac{\theta}{2}}$$

$$\text{Now, } \tan \theta = \frac{3}{4}$$

$$\Rightarrow \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} = \frac{3}{4}$$

$$3 - 3 \tan^2 \frac{\theta}{2} = 8 \tan \frac{\theta}{2}$$

$$\Rightarrow 3 \tan^2 \frac{\theta}{2} + 8 \tan \frac{\theta}{2} - 3 = 0$$

$$\Rightarrow \left(3 \tan \frac{\theta}{2} - 1 \right) \left(\tan \frac{\theta}{2} + 3 \right) = 0$$

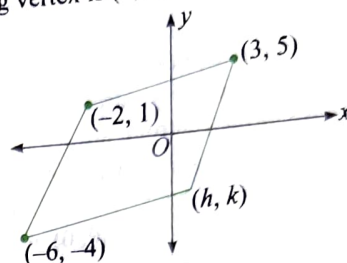
$$\text{So, } \tan \frac{\theta}{2} = \frac{1}{3}$$

$$\therefore \text{Slope of } OB = \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{2}$$

Multiple Correct Answers Type

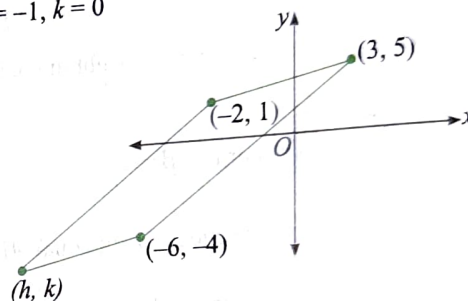
1. (2), (3), (4)

If the remaining vertex is (h, k) , then



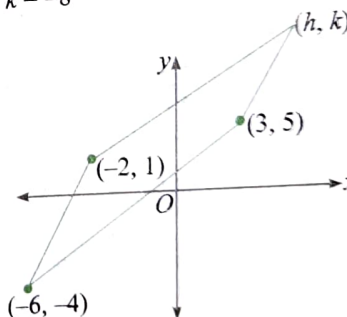
$$h - 2 = -6 + 3, k + 1 = 5 - 4$$

$$\text{or } h = -1, k = 0$$



$$h + 3 = -6 - 2, k + 5 = -4 + 1$$

$$\text{or } h = -11, k = -8$$



$$h - 6 = 3 - 2, k - 4 = 5 + 1$$

$$\text{or } h = 7, k = 10$$

2. (1), (4)

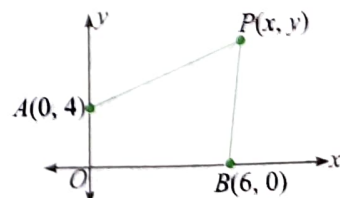
Let point P be (x, y) .

$$\text{ar } \triangle POA = \frac{1}{2} (OA) |x| = 2|x|$$

$$\text{ar } \triangle POB = \frac{1}{2} (OB) |y| = 3|y|$$

$$\therefore 2|x| = 6|y|$$

$$\text{or } |x| = 3|y| \text{ or } 3y - x = 0 \text{ or } 3y + x = 0$$



3. (3), (4)

Let the third vertex be (x, y) . Then,

$$\frac{1}{2} \begin{vmatrix} x & y & 1 \\ -4 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \pm 4$$

$$\Rightarrow x + 5y + 4 = \pm 8$$

$$\Rightarrow x + 5y + 12 = 0 \text{ or } x + 5y - 4 = 0$$

Hence, the third vertex lies on $x + 5y + 12 = 0$ or $x + 5y - 4 = 0$.

4. (2), (4)

Let any point on the line $x - y = 2$ be $C(h, h - 2)$. The given area of $\triangle ABC$ is

$$\left| \frac{1}{2} \begin{vmatrix} h & h-2 & 1 \\ -5 & 0 & 1 \\ 3 & 0 & 1 \end{vmatrix} \right| = 20$$

or $|8(h - 2)| = 40$

or $h - 2 = \pm 5$

or $h = 7, -3$

Hence, the points are $(7, 5)$ and $(-3, -5)$.

5. (1), (2)

Vertices $(a \cos \theta_1, a \sin \theta_1)$, $(a \cos \theta_2, a \sin \theta_2)$, and $(a \cos \theta_3, a \sin \theta_3)$ are equidistant from the origin $(0, 0)$. Hence, the origin is the circumcenter (centroid) of circumcircle. Therefore, the coordinates of the centroid are

$$\left(\frac{a(\cos \theta_1 + \cos \theta_2 + \cos \theta_3)}{3}, \frac{a(\sin \theta_1 + \sin \theta_2 + \sin \theta_3)}{3} \right)$$

But as the centroid is the origin $(0, 0)$, we have

$$\cos \theta_1 + \cos \theta_2 + \cos \theta_3 = 0$$

$$\text{and } \sin \theta_1 + \sin \theta_2 + \sin \theta_3 = 0$$

6. (1), (3), (4)

Since $AB = AC = 1$, the triangle is right-angled at point A . We have

$$\tan \alpha \tan \beta = -1$$

$$\text{or } \cos(\alpha - \beta) = 0 \text{ or } \alpha - \beta = \pm \frac{\pi}{2}$$

7. (1), (2)

For square $ABCD$, the vertices of diagonal AD , $A(2, -3)$ and $D(-1, 1)$, are given.

Let one of the other two vertices be $B(x, y)$. Now,

$$AB \perp BD$$

$$\therefore \left(\frac{y+3}{x-2} \right) \left(\frac{y-1}{x+1} \right) = -1$$

$$\text{or } y^2 + 2y - 3 = -x^2 + x + 2$$

$$\text{or } x^2 + y^2 - x + 2y - 5 = 0 \quad (1)$$

$$\text{Also, } AB = BD$$

$$\therefore (x-2)^2 + (y+3)^2 = (x+1)^2 + (y-1)^2$$

$$\text{or } 6x - 8y - 11 = 0 \quad (2)$$

Solving (1) and (2), we get two points as $(-3/2, -5/2)$ and $(5/2, 1/2)$.

8. (1), (3)

9. (1), (2), (3), (4)

The point $A(\alpha, \beta)$ lies on $y = 2x + 3$. Hence,

$$\beta = 2\alpha + 3$$

$$A \equiv (\alpha, 2\alpha + 3)$$

$$\text{Area of } \triangle ABC \text{ is } \left| \frac{1}{2} \begin{vmatrix} \alpha & 2\alpha+3 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} \right|$$

$$= \left| \frac{1}{2} [\alpha(2-3) + (2\alpha+3)(2-1) + 1(3-4)] \right|$$

$$= \frac{1}{2} |-\alpha + 2\alpha + 3 - 1| = \frac{1}{2} |\alpha + 2| = S$$

$$[S] = 2 \text{ or } 2 \leq S < 3$$

$$\therefore 2 \leq \frac{1}{2} |\alpha + 2| < 3$$

$$4 \leq |\alpha + 2| < 6$$

$$|\alpha + 2| < 6$$

$$\text{or } -6 < \alpha + 2 < 6$$

$$\text{or } -8 < \alpha < 4$$

$$\text{and } |\alpha + 2| \geq 4$$

$$\text{i.e., } \alpha + 2 \geq 4 \text{ or } \alpha + 2 \leq -4$$

$$\text{i.e., } \alpha \geq 2 \text{ or } \alpha \leq -6$$

(2)

From (1) and (2), we get

$$-8 < \alpha \leq -6 \text{ or } 2 \leq \alpha < 4$$

$$\text{or } \alpha = -7, -6, 2, 3$$

The possible coordinates of A are $(-7, -11)$, $(-6, -9)$, $(2, 7)$, and $(3, 9)$.

10. (1), (2), (3), (4)

As H (orthocenter), G (centroid), and C (circumcenter) are collinear, we have

$$\begin{vmatrix} 4 & b & 1 \\ b & 2b-8 & 1 \\ -4 & 8 & 1 \end{vmatrix} = 0$$

$$\text{or } \begin{vmatrix} 4 & b & 1 \\ b-4 & b-8 & 0 \\ -(b+4) & 16-2b & 0 \end{vmatrix} = 0$$

$$\text{or } (b-4)(16-2b) + (b+4)(b-8) = 0$$

$$\text{or } 2(b-4)(8-b) + (b+4)(b-8) = 0$$

$$\text{or } (8-b)[(2b-8) - (b+4)] = 0$$

$$\text{or } (8-b)(b-12) = 0$$

Hence, $b = 8$ or 12 , which is wrong because collinearity does not explain centroid, orthocenter, and circumcenter.

Now, H, G , and C are collinear and $HG/GC = 2$. Therefore,

$$\frac{-8+4}{3} = b \text{ or } b = \frac{-4}{3}$$

$$\text{and } \frac{16+b}{3} = 2b-8 \text{ or } b = 8$$

But no common value of b is possible.

11. (1), (3)

$$S = \text{Area of } \triangle OAC + \text{Area of } \triangle BCD$$

$$= \frac{1 \cdot x}{2} + \frac{(1-x)(y-1)}{2}, 0 < x < 1$$

$$\therefore S = \frac{x}{2} - \frac{(x-1)(y-1)}{2} \quad (1)$$

Now, $\triangle$'s BCD and OCA are similar. Therefore,

$$\frac{y-1}{1} = \frac{1-x}{x}$$

$$\text{or } y = 1 + \frac{1-x}{x} = \frac{1}{x}$$

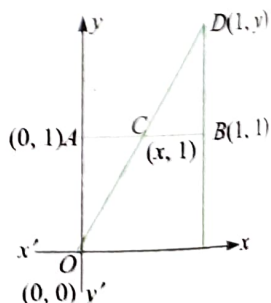
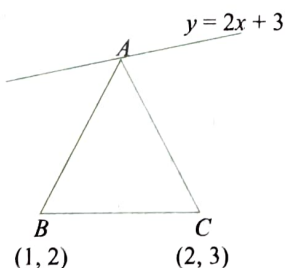
$$\therefore S = \frac{x}{2} - \frac{(x-1)\{(1/x) - 1\}}{2}$$

$$= \frac{x}{2} + \frac{(x-1)^2}{2x}$$

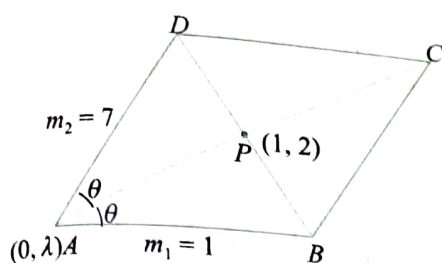
$$= \frac{x^2 + (x-1)^2}{2x} = \frac{2x^2 - 2x + 1}{2x}$$

$$= x + \frac{1}{2x} - 1 = \left(\sqrt{x} - \frac{1}{\sqrt{2x}} \right)^2 - 1 + \sqrt{2}$$

Therefore, A is minimum if $\sqrt{x} = \frac{1}{\sqrt{2x}}$. Therefore, $x = \frac{1}{\sqrt{2}}$ which lies in $(2/3, 1)$ and $A_{\min} = \sqrt{2} - 1$ which lies in $(1/3, 1/2)$.



12. (2), (3)



Let point A be $(0, \lambda)$.

$$\text{Slope of } AP = \frac{\lambda - 2}{0 - 1} = 2 - \lambda$$

AB and AC are parallel to the lines $y = x + 2$ and $y = 7x + 3$, respectively.

Thus, slope of AB and AC are 1 and 7, respectively.

From the figure,

$$\tan \theta = \frac{(2 - \lambda) - 1}{1 + (2 - \lambda)} = \frac{7 - (2 - \lambda)}{1 + 7(2 - \lambda)}$$

$$\Rightarrow \frac{1 - \lambda}{3 - \lambda} = \frac{5 + \lambda}{15 - 7\lambda}$$

$$\Rightarrow (\lambda - 1)(7\lambda - 15) = (5 + \lambda)(3 - \lambda)$$

$$\Rightarrow 8\lambda^2 - 20\lambda = 0$$

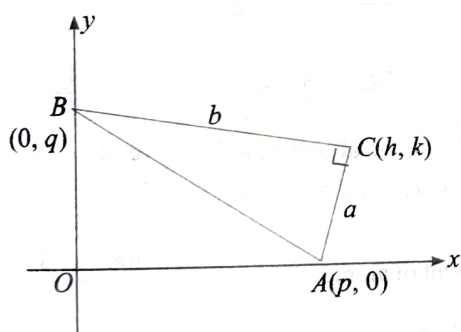
$$\therefore \lambda = 0, 5/2$$

$$A = (0, 0) \text{ and } (0, 5/2)$$

13. (2), (4)

Let the coordinates of A and B be, respectively, $(p, 0)$ and $(0, q)$.

Let C be (h, k) .



From the figure, we have

$$(h - p)^2 + k^2 = a^2 \quad (1)$$

$$h^2 + (k - q)^2 = b^2 \quad (2)$$

$$p^2 + q^2 = a^2 + b^2 \quad (3)$$

Subtracting (2) from (1), we get

$$2qk - 2ph = a^2 - b^2$$

Solving this with (1), we get

$$p = \left[h \pm \sqrt{a^2 - k^2} \right]$$

$$\text{and } q = k \pm \sqrt{b^2 - h^2}$$

Putting these values in (3), we get

$$\left[h \pm \sqrt{a^2 - k^2} \right]^2 + \left[k \pm \sqrt{b^2 - h^2} \right]^2 = a^2 + b^2$$

$$\Rightarrow \pm 2h \sqrt{a^2 - k^2} \pm 2k \sqrt{b^2 - h^2} = 0$$

$$\Rightarrow h^2 a^2 = k^2 b^2$$

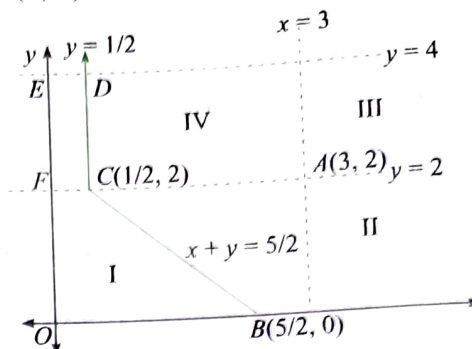
Therefore, locus of C is $ax \pm by = 0$.

Linked Comprehension Type

For Problems 1-3

Let $P = (h, k)$ be a point in the first quadrant such that

$$d(P, A) = d(P, O)$$



$$\therefore |h - 3| + |k - 2| = |h| + |k| = h + k \quad (1)$$

[h and k are +ve, point $P(h, k)$ being in the first quadrant]

If $h < 3$ and $k < 2$, then (h, k) lies in region I. Then,

$$3 - h + 2 - k = h + k$$

$$\text{or } h + k = \frac{5}{2}$$

If $h > 3$ and $k < 2$, then (h, k) lies in region II. Then,

$$h - 3 + 2 - k = h + k$$

$$\text{or } k = -\frac{1}{2} \text{ (not possible)}$$

If $h > 3$ and $k > 2$, then (h, k) lies in region III. Then,

$$h - 3 + k - 2 = h + k$$

$$\text{or } -5 = 0 \text{ (not possible)}$$

If $h < 3$ and $k > 2$, then (h, k) lies in region IV. Then,

$$3 - h + k - 2 = h + k \text{ or } h = \frac{1}{2}$$

Hence, the required set consists of line segment $x + y = 5/2$ of finite length as shown in region I and ray $x = 1/2$ in region IV.

1. (4) Obviously, the locus of P is the union of line segment and one infinite ray.

2. (2) Area of region OBCDEFO

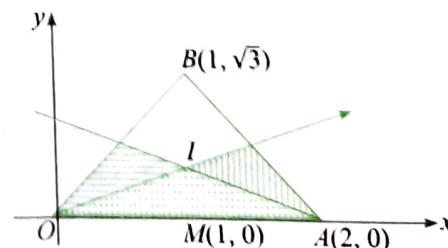
$$A = \text{Area of trapezium OBCF} + \text{Area of rectangle FCDE}$$

$$= \frac{1}{2} \times \left(\frac{5}{2} + \frac{1}{2} \right) \times 2 + \frac{1}{2} \times 2$$

$$= 4$$

3. (4) Obviously, the locus of P is a relation but not a function.

4. (4)



$$d(P, OA) \leq \min[d(P, OB), d(P, AB)]$$

$$\text{or } d(P, OA) \leq d(P, OB)$$

$$\text{and } d(P, OA) \leq d(P, AB)$$

When $d(P, OA) = d(P, OB)$, P is equidistant from OA and OB , or P lies on the angle bisector of lines OA and OB .

Hence, when $d(P, OA) \leq d(P, OB)$, point P is nearer to OA than to OB or lies below the angle bisector of $\angle AOB$. Similarly, when $d(P, OA) \leq d(P, AB)$, P is nearer to OA than to AB , or P lies below the bisector of $\angle OAB$ and AB . Therefore, the required area is equal to the area of ΔOIA . Now,

$$\tan \angle BOA = \frac{\sqrt{3}}{1} = \sqrt{3}$$

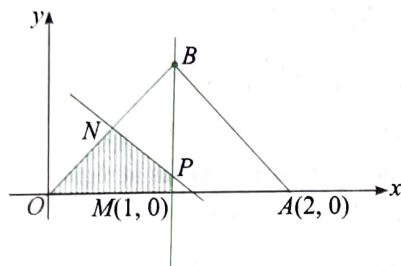
or $\angle BOA = 60^\circ$

Hence, the triangle is equilateral. Then I coincides with the centroid, which is $(1, 1/\sqrt{3})$.

Therefore, the area of ΔOIA is

$$\frac{1}{2} OA \times IM = \frac{1}{2} \times 2 \times \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ sq. units}$$

5. (2)



$$OP \leq \min[BP, AP]$$

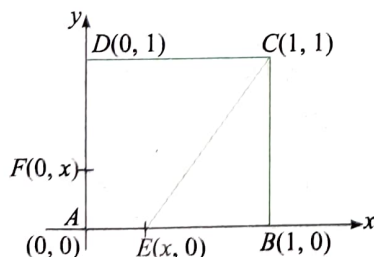
or $OP \leq BP$ (when $BP < AP$)

$$OP \leq AP$$
 (when $AP < BP$)

Let $OP = BP$. Then P lies on the perpendicular bisector of OB . For $OP = AP$, P lies on the perpendicular bisector of OA . Then, for the required condition, P lies in the region as shown in the diagram. The area of region $OMPN$ is

$$\begin{aligned} \frac{1}{2} \times \begin{vmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1/\sqrt{3} \\ 1 & \sqrt{3}/2 \\ 0 & 0 \end{vmatrix} &= \frac{1}{2} \left[\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}} \right] \\ &= \frac{1}{2} \left[\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \right] \\ &= \frac{1}{\sqrt{3}} \end{aligned}$$

6. (3)



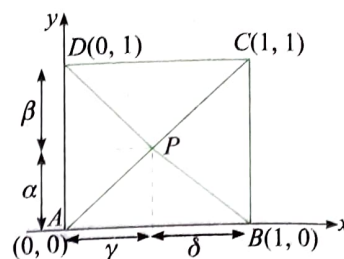
Area of $CDFE$

$$A = 1 - \frac{1}{2}x^2 - \frac{1}{2}(1-x)$$

$$= \frac{2-x^2-1+x}{2} = \frac{1+x-x^2}{2} = \frac{5}{4} - \left(x - \frac{1}{2}\right)^2$$

$$A_{\max} = \frac{5}{8} \text{ at } x = \frac{1}{2}$$

7. (4)

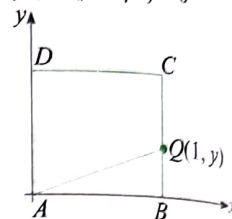


$$\begin{aligned} (PA)^2 - (PB)^2 + (PC)^2 - (PD)^2 \\ = (\alpha^2 + \gamma^2) - (\alpha^2 + \delta^2) + (\delta^2 + \beta^2) - (\gamma^2 + \beta^2) = 0 \end{aligned}$$

8. (2) $\frac{1}{2} y(1) = \frac{1}{3}(1)$

or $y = \frac{2}{3}$

$$\therefore AQ = \sqrt{(1)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{13}}{3}$$



9. (2) Let the coordinates of D be (α, β) . Then,

$$\frac{\alpha + 1 + 3}{3} = 3 \text{ or } \alpha = 5$$

and $\frac{\beta + 2 + 4}{3} = 2 \text{ or } \beta = 0$

$$\therefore D \equiv (5, 0)$$

Taking $A(x_1, y_1)$, $B(x_2, y_2)$, and $C(x_3, y_3)$, we have

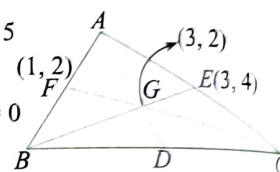
$$\frac{x_1 + x_2}{2} = 1,$$

$$\frac{x_2 + x_3}{2} = 5, \frac{x_3 + x_1}{2} = 3$$

and $\frac{y_1 + y_2}{2} = 2, \frac{y_2 + y_3}{2} = 0, \frac{y_1 + y_3}{2} = 4$

Solving the above equations, we get

$$A \equiv (-1, 6), B \equiv (3, -2), C \equiv (7, 2)$$

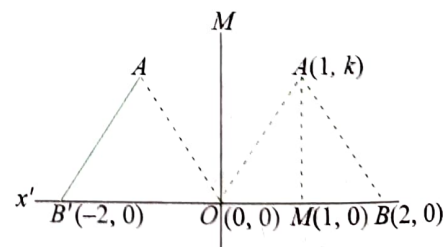


10. (3) Height of altitude from $A = \frac{2 \times \text{area}(\Delta ABC)}{BC} = 6\sqrt{2}$

Matrix Match Type

1. $a \rightarrow p$; $b \rightarrow p, s$; $c \rightarrow p, r$; $d \rightarrow q, s$.

a.



From the figure, ΔABC is equilateral. Hence,

$$\tan 60^\circ = k$$

i.e., $k = \sqrt{3}$ (for first quadrant)

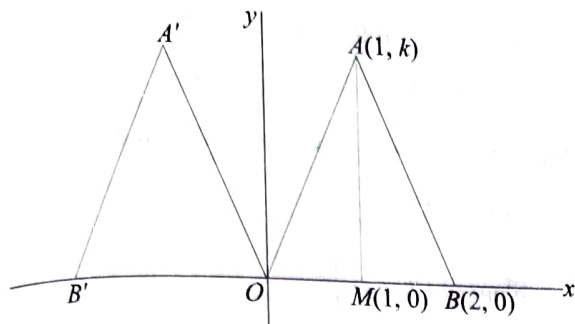
or $k = -\sqrt{3}$ (for fourth quadrant)

Then the possible coordinates are $(1, \pm\sqrt{3})$.

Similarly, for the second quadrant, the point is $(-1, \sqrt{3})$ and for the third quadrant, the point is $(-1, -\sqrt{3})$.

b. Case i:

If $OA = AB$, then $\angle A = 30^\circ$. Therefore,



$$\angle AOB = 75^\circ$$

$$\frac{AM}{OM} = \tan 75^\circ$$

$$AM = OM \tan 75^\circ$$

$$k = 1 \times (2 + \sqrt{3})$$

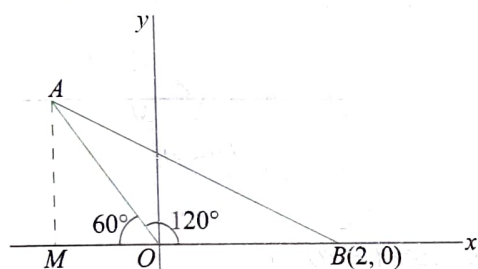
$$k = 2 + \sqrt{3}$$

Hence, point A is $(1, 2 + \sqrt{3})$. By symmetry, all possible points are $(\pm 1, \pm(2 + \sqrt{3}))$.

Case ii:

$$AO = OB$$

$$\angle AOB = 120^\circ$$

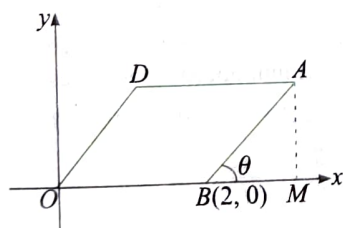


$$AM = 2 \sin 60^\circ = \sqrt{3}$$

$$\text{and } OM = 2 \cos 60^\circ = 1$$

Hence, point A is $(-1, -\sqrt{3})$. By symmetry, all possible points are $(\pm 1, \pm\sqrt{3})$.

c.



Let $\angle DOB = \angle ABM = \theta$.

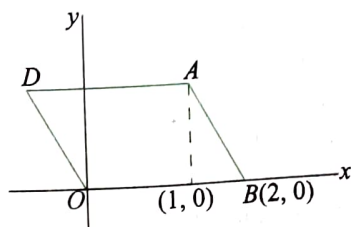
$$\text{Area of } \triangle OAB = \frac{1}{2} \times OB \times AM = \frac{1}{2} \times 2\sqrt{3}$$

$$\text{or } 2 \times 2 \sin \theta = 2\sqrt{3}$$

$$\text{or } \sin \theta = \frac{\sqrt{3}}{2}$$

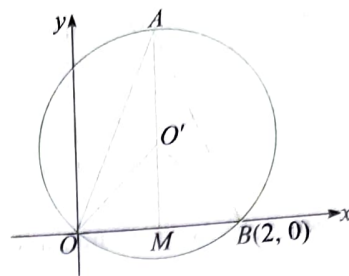
$$\text{or } AM = \sqrt{3} \text{ and } BM = 1$$

Hence, A has coordinates $(3, \sqrt{3})$. By symmetry, all the possible coordinates are $(\pm 3, \pm\sqrt{3})$.



From the figure, A has coordinates $(1, \sqrt{3})$.
By symmetry, all possible coordinates are $(\pm 1, \pm\sqrt{3})$.

d.



$$OB = 2 \text{ units} = OO' = \text{Radius}$$

$$\text{or } OM = \frac{2}{2} = 1 \text{ unit}$$

In $\triangle OO'M$,

$$O'M = \sqrt{4-1} = \sqrt{3}$$

Since $\triangle OAB$ is isosceles, point A lies on the perpendicular bisector of OB . Therefore,

$$AM = \sqrt{3} + 2 = O'M + OA$$

Hence, the coordinates of A will be $(1, 2 + \sqrt{3})$ in the first quadrant. By symmetry, all possible coordinates of A are $(\pm 1, \pm(2 + \sqrt{3}))$.

2. a $\rightarrow$ s; b $\rightarrow$ p; c $\rightarrow$ q; d $\rightarrow$ r.

$$\text{a. Centroid is } \left(\frac{0+5+16}{3}, \frac{0+12+12}{3} \right) \equiv (7, 8)$$

b. Let the coordinates of circumcenter be $O(x, y)$.
Therefore,

$$OA = OB = OC$$

$$\therefore x^2 + y^2 = (x-5)^2 + (y-12)^2$$

$$= (x-16)^2 + (y-12)^2$$

$$\therefore x^2 + y^2 = (x-5)^2 + (y-12)^2$$

$$\text{or } 10x + 24y = 169$$

$$\text{and } (x-5)^2 + (y-12)^2 = (x-16)^2 + (y-12)^2$$

$$\text{or } 2x = 21$$

Solving, we get

$$x = \frac{21}{2}, y = \frac{8}{3}$$

$$\text{c. } I \equiv \left(\frac{0 \times 11 + 5 \times 20 + 16 \times 13}{13 + 20 + 11}, \frac{0 \times 11 + 12 \times 20 + 13 \times 12}{13 + 20 + 11} \right) \equiv (7, 9)$$

$$\text{d. } I_2 \equiv \left(\frac{-5 \times 20 + 13 \times 16 + 11 \times 0}{-20 + 13 + 11}, \frac{-12 \times 20 + 0 \times 11 + 13 \times 12}{-20 + 13 + 11} \right) \equiv (27, -21)$$

3. a $\rightarrow$ r; b $\rightarrow$ s; c $\rightarrow$ q; d $\rightarrow$ p.

Consider the points $P(x, y)$, $A(0, -4)$, and $B(3, 0)$. Now,

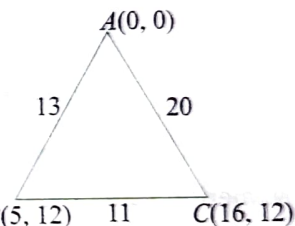
$$\sqrt{x^2 + y^2 + 8y + 16} - \sqrt{x^2 + y^2 - 6x + 9} = 5$$

$$\text{or } \sqrt{x^2 + (y+4)^2} - \sqrt{(x-3)^2 + y^2} = 5$$

$$\text{or } AP - BP = AB$$

Thus, the locus of P is an infinite ray.

$$\sqrt{x^2 + y^2 + 8y + 16} - \sqrt{x^2 + y^2 - 6x + 9} = \pm 5$$



Therefore, the locus of point P is two rays in opposite direction.

$$\sqrt{x^2 + y^2 + 8y + 16} + \sqrt{x^2 + y^2 - 6x + 9} = 5$$

or $AP + BP = AB$

Therefore, the locus of point P is line segment AB .

$$\sqrt{x^2 + y^2 + 8y + 16} - \sqrt{x^2 + y^2 - 6x + 9} = 7$$

or $AP - BP = 7 > AB$

which is not possible. Hence, no locus.

4. (2)

- a. Feet of perpendicular drawn from a point $P(-3, 4)$ on axes are $A(-3, 0)$ and $B(0, 4)$.

$\therefore AB = 5$

- b. Distance of $A(3, -2)$ from y -axis is 2.

So, the length of altitude of triangle from vertex A is 6.

$\therefore$ Length of side of triangle = $\frac{6}{\sin 60^\circ} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$

- c. The collinear points do not form a triangle. Hence area is 0 sq. unit.

- d. Points $A(-2, 0)$, $B(-1, \frac{1}{\sqrt{3}})$ and $C(\cos \theta, \sin \theta)$ are collinear.

$\therefore$ Slope of AB = Slope of AC

$$\therefore \frac{\frac{1}{\sqrt{3}}}{-1+2} = \frac{\sin \theta}{\cos \theta + 2}$$

$$\therefore \sqrt{3} \sin \theta - \cos \theta = 2$$

$$\therefore \sin \left(\theta - \frac{\pi}{6} \right) = 1$$

Therefore, only one value of θ is there for $\theta \in [0, 2\pi]$.

Numerical Value Type

1. (220) Area of $\triangle AOB$, $\frac{ab}{2} = 11$

$\therefore ab = 22$

But points $A(a, 0)$, $B(0, b)$ and $P(2, 3)$ are collinear.

$\Rightarrow$ Slope of AB = Slope of AP

$$\Rightarrow \frac{b}{a} = \frac{3-0}{2-a}$$

$$\Rightarrow -2b + ab = 3a$$

$$\Rightarrow 3a + 2b = 22$$

Squaring $9a^2 + 4b^2 + 12ab = 484$

$$\Rightarrow 9a^2 + 4b^2 = 220$$

2. (7) Using section formula, $A \equiv \left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1} \right)$.

Area of triangle ABC is 2 sq. units. Therefore,

$$\frac{1}{2} \begin{vmatrix} 1 & 5 & 1 \\ 7 & -2 & 1 \\ \frac{3k-5}{k+1} & \frac{5k+1}{k+1} & 1 \end{vmatrix} = \pm 2$$

Operating $R_2 \rightarrow R_2 - R_1$; $R_3 \rightarrow R_3 - R_1$, we get

$$\begin{vmatrix} 1 & 5 & 1 \\ 6 & -7 & 0 \\ \frac{3k-5}{k+1} - 1 & \frac{5k+1}{k+1} - 5 & 0 \end{vmatrix} = \pm 4$$

$$\text{or } 6 \left(\frac{5k+1-5k-5}{k+1} \right) + 7 \left(\frac{3k-5-k-1}{k+1} \right) = \pm 4$$

$$\text{or } -24 + 7(2k-6) = \pm 4(k+1)$$

$$\text{i.e., } k = 7 \text{ or } k = \frac{31}{9}$$

3. (5) Given vertices of triangle are $O(0, 0)$, $B(6, 8)$, and $C(-4, 3)$.

Slope of $OB = \frac{8}{6} = \frac{4}{3}$

Slope of $OC = -\frac{3}{4}$

$\therefore \angle BOC = \frac{\pi}{2}$

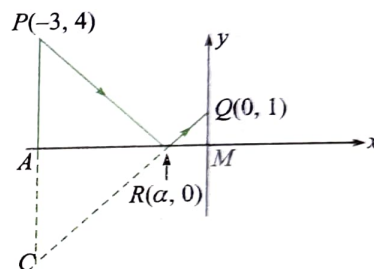
$\triangle OBC$ is right-angled at O .

Circumcenter = Midpoint of hypotenuse $BC = \left(1, \frac{11}{2} \right)$

Orthocenter = Vertex $O(0, 0)$

Required distance = $\sqrt{\left(1 + \frac{121}{4} \right)} = \frac{5\sqrt{5}}{2}$ units

4. (0.6) For $PR = RQ$ to be minimum, it should be the path of light.



$\therefore \angle PRA = \angle QRM$

From similar triangles PAR and QMR ,

$$\frac{AR}{RM} = \frac{PA}{QM}$$

$$\text{or } \frac{\alpha+3}{0-\alpha} = \frac{4}{1} \text{ or } \alpha = -\frac{3}{5}$$

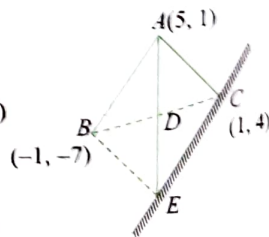
5. (4) $(AP + PC)$ is minimized when P lies on AC and $BP + PD$ is minimized when P lies on BD . Thus,

$$AP + PB + PC + PD = AC + BD = 2\sqrt{5}$$

6. (37.5) $\frac{AD}{DE} = \frac{BD}{DC} = \frac{AB}{AC} = 2$

$\therefore D \equiv \left(\frac{1}{3}, \frac{1}{3} \right)$ and $E \equiv (-2, 0)$

Area of $\triangle ACE$ + Area of $\triangle ABC$
 $= \frac{25}{2} + 25 = 37.5$ sq. units



7. (8) We know that the area of the triangle formed by joining the midpoints of any triangle is one-fourth of that triangle. Therefore,

Required area = 8

8. (2) Let $m = \tan 2\theta$ and $n = \tan \theta$

And $m = 4n$

and $m \neq 0$,

$\therefore \tan 2\theta \neq 0$;

$\therefore \theta \neq 0$

$\tan 2\theta = 4 \tan \theta$

$$\Rightarrow \frac{2 \tan \theta}{1 - \tan^2 \theta} = 4 \tan \theta$$

$$\Rightarrow 1 = 2 - 2 \tan^2 \theta$$

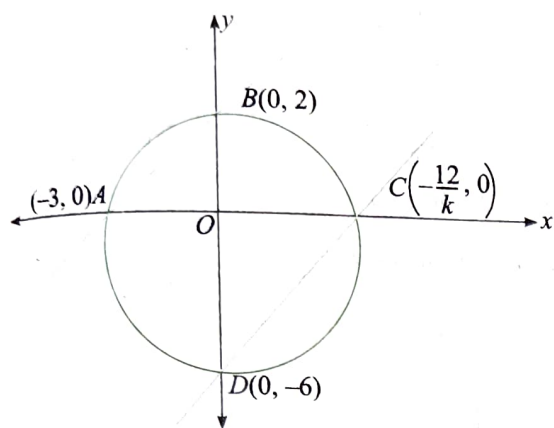
$$\Rightarrow 2 \tan^2 \theta = 1$$

$$\Rightarrow \tan^2 \theta = 1/2$$

$$\Rightarrow mn = \tan 2\theta \cdot \tan \theta$$

$$= \frac{2 \tan^2 \theta}{1 - \tan^2 \theta} = \frac{1}{1 - (1/2)} = 2$$

11. (3)



$2x - 3y + 6 = 0$ meets the axes at $A(-3, 0)$ and $B(0, 2)$.

$kx + 2y + 12 = 0$ meets the axes at $D(0, -6)$ and $C(-12/k, 0)$.

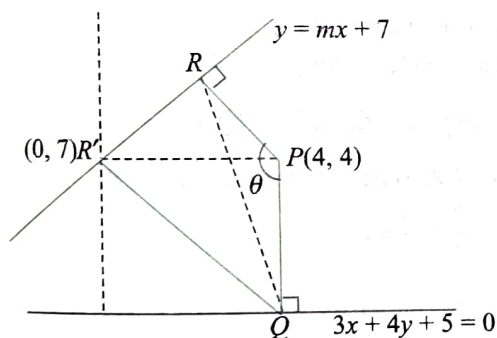
Points A, B, C and D will be concyclic, if

$$OA \times OC = OB \times OD$$

$$\Rightarrow 3 \times (-12/k) = 2 \times 6$$

$$\Rightarrow k = -3$$

11. (4)



Since PQ is of fixed length,

$$\text{Area of } \triangle PQR = \frac{1}{2} |PQ| |RP| \sin \theta$$

This will be maximum if $\sin \theta = 1$ and RP is maximum.

Since $y = mx + 7$ passes through $R'(0, 7)$, it must be rotated about $(0, 7)$ such that in new position, θ becomes 90° i.e., the line becomes perpendicular to $3x + 4y + 5 = 0$. Then the slope of line is $\frac{4}{3}$.

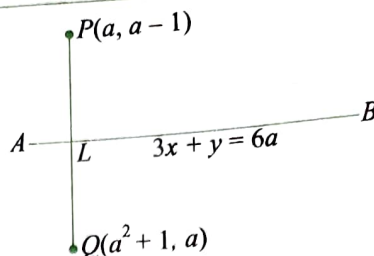
$$\text{Also, Slope of } PR' = \frac{4-7}{4-0} = -\frac{3}{4} = \text{Slope of given line}$$

That is the in new position of the line, R coincides with R' .

11. (2) Image of point P in line AB is Q .

Thus, L is the midpoint of AB .

$$\therefore L \equiv \left(\frac{a^2 + a + 1}{2}, \frac{2a - 1}{2} \right)$$



Since L lies on line AB ,

$$\therefore 3 \left(\frac{a^2 + a + 1}{2} \right) + \frac{2a - 1}{2} = 6a$$

$$\Rightarrow 3a^2 - 7a + 2 = 0$$

$$\therefore a = 2, \frac{1}{3}$$

Also, $PQ \perp AB$.

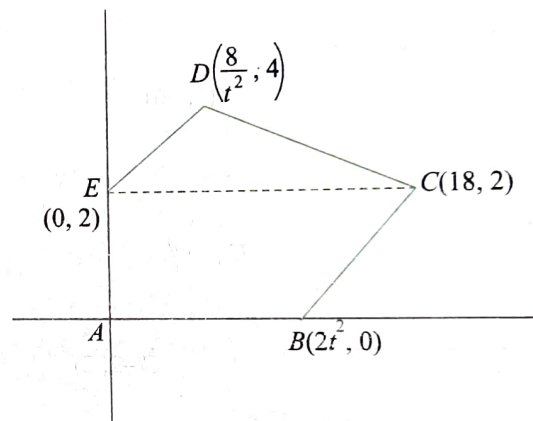
$$\Rightarrow -3 \times \frac{1}{a^2 - a + 1} = -1$$

$$\Rightarrow a^2 - a - 2 = 0$$

$$\therefore a = 2, -1$$

Therefore, common value of a is 2.

12. (54)



$$\text{Slope of } BC = \frac{2-0}{18-2t^2} \geq 0$$

$$\Rightarrow 18 - 2t^2 \geq 0$$

$$\Rightarrow t^2 - 9 \leq 0 \Rightarrow t \in [-3, 3]$$

Now, area of quadrilateral

$$= \text{Area of trapezium } ABCE + \text{Area of triangle } CDE$$

$$= \frac{1}{2} (2t^2 + 18) \times 2 + \frac{1}{2} \times 18 \times 2$$

$$= 36 + 2t^2 \leq 54$$

So, the required maximum area is 54 sq. units.

Archives

JEE MAIN

Single Correct Answer Type

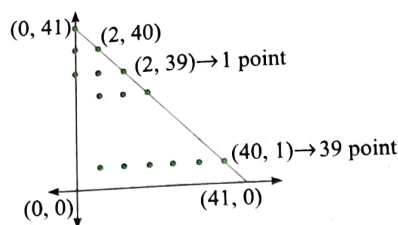
1. (2) The lines must be parallel; therefore, slopes are equal. So $p(p^2 + 1) = -(p^2 + 1) \Rightarrow p = -1$

2. (3) $\therefore C \left(\frac{8}{5}, \frac{14}{5} \right)$, which lies on $2x + y = k$

$$\therefore \frac{2 \times 8}{5} + \frac{14}{5} = k$$

$$k = 6$$

3. (4)



$$\begin{aligned}\text{So number of points} &= 1 + 2 + 3 + \dots + 39 \\ &= \frac{39 \times 40}{2} \\ &= 780\end{aligned}$$

4. (1) We have

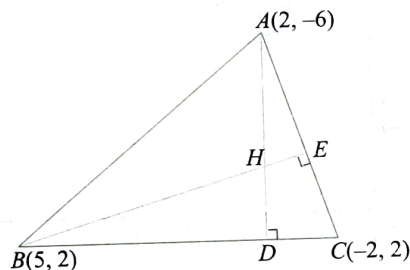
$$\frac{1}{2} \begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix} = \pm 28$$

$$\Rightarrow 5k^2 + 13k - 46 = 0$$

$$\text{or } 5k^2 + 13k + 66 = 0 \text{ (no real solution)}$$

$$\Rightarrow k = \frac{-23}{5} \text{ or } k = 2$$

Since k is an integer, $k = 2$.



From the figure, slope of BC is 0.

So, equation of AD is

$$x = 2$$

Slope of AC is -2 .

So, slope of BE is $1/2$.

Equation of line BE is

$$x - 2y - 1 = 0$$

(1)

(2)

Solving equations (1) and (2), we get orthocentre at point $(2, 1/2)$.

5. (4) In a triangle, orthocenter (A), centroid (B) and circumcentre (C) are always collinear such that $AB : BC = 2 : 1$.



So, slope of AD is infinity.

Let orthocenter have coordinates $(2, k)$.

Slope of AC is -2 .

$$\therefore \text{Slope of } BH, \frac{k-2}{2-5} = \frac{1}{2}$$

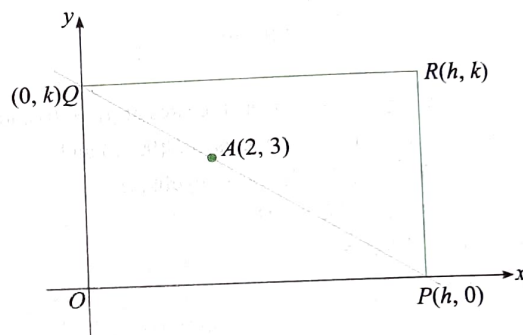
$$\therefore k = \frac{1}{2}$$

$$\therefore AC = \sqrt{(6-(-3))^2 + (2-5)^2} = \sqrt{81+9} = \sqrt{90} = 3\sqrt{10}$$

AC is diameter of the circle.

$$\therefore \text{Radius of circle} = \frac{3}{2}\sqrt{10} = 3\sqrt{\frac{5}{2}}$$

6. (4)



Let the point R be (h, k) .

Thus, $P \equiv (h, 0)$ and $Q \equiv (0, k)$

Equation of variable line PQ is $\frac{x}{h} + \frac{y}{k} = 1$.

Point $A(2, 3)$ lies on the variable line PQ .

$$\therefore \frac{2}{h} + \frac{3}{k} = 1$$

$$\text{or } \frac{2}{x} + \frac{3}{y} = 1$$

Therefore, the locus of point R is $2y + 3x = xy$.

Chapter 2

Concept Application Exercises

Exercise 2.1

1. The right bisector of a line segment bisects the line segment at 90° .
The endpoints of the line segment are given as $A(3, 4)$ and $B(-1, 2)$.
Accordingly, the midpoint of AB is

$$\left(\frac{3-1}{2}, \frac{4+2}{2}\right) \equiv (1, 3)$$

$$\text{Slope of } AB = \frac{2-4}{-1-3} = \frac{-2}{-4} = \frac{1}{2}$$

$$\therefore \text{Slope of line perpendicular to } AB = -\frac{1}{(1/2)} = -2$$

The equation of the line passing through $(1, 3)$ and having a slope of -2 is

$$(y-3) = -2(x-1)$$

$$\text{or } 2x + y = 5$$

2. Midpoint of $AB = E\left(\frac{a+a'}{2}, \frac{b+b'}{2}\right)$ and Midpoint of $CD = F\left(\frac{a'-a}{2}, \frac{b-b'}{2}\right)$

Hence, the equation of line EF is

$$y - \frac{b+b'}{2} = \frac{b-b'-b-b'}{a'-a-a-a'} \left(x - \frac{a+a'}{2}\right)$$

On simplification, we get

$$2ay - 2b'x = ab - a'b'$$

3. The required equation of median is

$$y + 8 = \frac{-(3/2) + 8}{-2 - 5}(x - 5)$$

$$\text{or } 13x + 14y + 47 = 0$$

4. The given line is

$$bx - ay = ab$$

Obviously, it cuts the x -axis at $(a, 0)$.

The equation of line perpendicular to above line is

$$ax + by = k$$

But it passes through $(a, 0)$. Therefore,

$$k = a^2$$

Hence, the required equation of line is

$$ax + by = a^2$$

$$\text{i.e., } \frac{x}{a} + \frac{y}{b} = \frac{a}{b}$$

5. Slope of $DE = \frac{7-3}{5-1} = 1$

$$\text{Slope of } AB = 1$$

Hence, the equation of AB is

$$y - 7 = x + 5$$

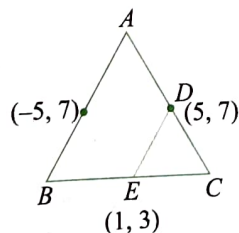
$$\text{or } x - y + 12 = 0$$

6. Angle between both the lines is

$$\tan^{-1}m \pm \tan^{-1}m = \tan^{-1}\frac{2m}{1-m^2} \text{ or } \tan^{-1}0$$

Therefore, the equations of the lines are

$$y = 0, y = \frac{2mx}{1-m^2}$$



7. The midpoint of $Q(-2, 6)$ and $P(4, 2)$ is

$$\left(\frac{-2+4}{2}, \frac{6+2}{2}\right), \text{ i.e., } (1, 4)$$

and the gradient of line PQ is

$$\frac{2-6}{4-2} = \frac{-2}{2} = -1$$

Therefore, the slope of L is $3/2$.

Hence, the equation of line which passes through point $(1, 4)$ is

$$y - 4 = \frac{3}{2}(x - 1)$$

$$\text{or } 3x - 2y + 5 = 0$$

8. $x + 2|y| = 1$ and $x = 0$ gives

$$|y| = \frac{1}{2}$$

$$\therefore y = \pm \frac{1}{2}$$

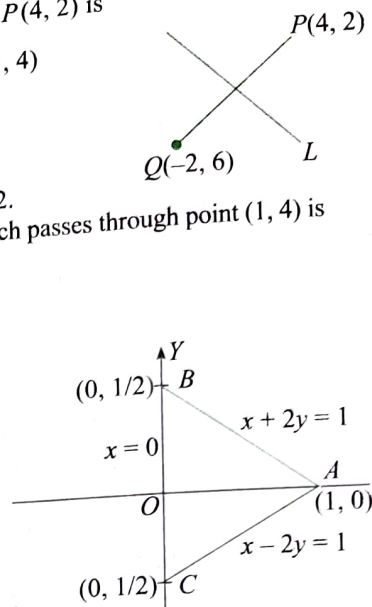
Required area

$$= \text{Area of } \triangle ABC$$

(where AB is $x + 2y = 1$ and

AC is $x - 2y = 1$)

$$\therefore \text{Required area} = \frac{1}{2}(BC)(OA) = \frac{1}{2}(1)(1) = \frac{1}{2}$$



9. Point of intersection of $x - 2y = 1$ and $x + 3y = 2$ is $(7/5, 1/5)$.

Any line parallel to $3x + 4y = 0$ is $3x + 4y + K = 0$.

As this line passes through $(7/5, 1/5)$, we get

$$\frac{21}{5} + \frac{4}{5} + K = 0$$

$$\text{or } K = -5$$

Therefore, the required line is

$$3x + 4y - 5 = 0$$

10. Let $P(3, -4)$ be the foot of the perpendicular from the origin O on the required line. Then the slope of OP is

$$\frac{-4-0}{3-0} = -\frac{4}{3}$$

Therefore, the slope of the required line is $3/4$.

Hence, its equation is

$$y + 4 = \frac{3}{4}(x - 3)$$

$$\text{or } 3x - 4y = 25$$

11. $\sqrt{3}x + y = 0$ makes an angle of 120° with OX , whereas $\sqrt{3}x - y = 0$ makes an angle of 60° with OX . Therefore, the required line is $y - 2 = 0$.

12. Let a and b be the intercepts on the axes. Then the equation of the line is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (1)$$

Since (1) passes through $(4, 3)$, we get

$$\frac{4}{a} + \frac{3}{b} = 1 \quad (2)$$

Also, given that

$$a + b = -1 \quad (3)$$

From (3), we get

$$b = -1 - a \quad (4)$$

Putting in (2), we get

$$\frac{4}{a} + \frac{3}{-1-a} = 1$$

$$\text{or } -4 - 4a + 3a = -a - a^2$$

or $a^2 = 4$

or $a = \pm 2$

When $a = 2$, then from (4), $b = -1 - 2 = -3$

and when $a = -2$, $b = -1 + 2 = 1$

Therefore, the line is

$$\frac{x}{2} + \frac{y}{-3} = 1 \text{ and } \frac{x}{-2} + \frac{y}{1} = 1$$

13. Let the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

This line meets the x -axis at $A(a, 0)$ and the y -axis at $B(0, b)$.

Therefore, we get

$$(3, 4) \equiv \left(\frac{a}{2}, \frac{b}{2}\right)$$

or $\frac{a}{2} = 3, \frac{b}{2} = 4$

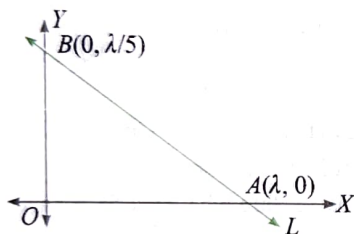
or $a = 6, b = 8$

Therefore, the required line is

$$\frac{x}{6} + \frac{y}{8} = 1$$

or $4x + 3y = 24$

14. The given line is $5x - y = 1$. Therefore, the equation of line L which is perpendicular to the given line is $x + 5y = \lambda$. This line meets the coordinate axes at $A(\lambda, 0)$ and $B(0, \lambda/5)$.



$\therefore$ Area of $\triangle OAB = \frac{1}{2} \times OA \times OB$

or $5 = \frac{1}{2} \times \lambda \times \frac{\lambda}{5}$

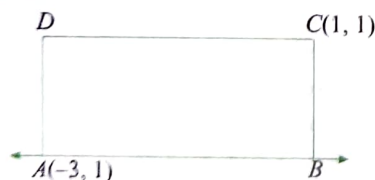
or $\lambda^2 = 5^2 \times 2$

or $\lambda = \pm 5\sqrt{2}$

Hence, the equation of line L is

$$x + 5y - 5\sqrt{2} = 0 \text{ or } x + 5y + 5\sqrt{2} = 0$$

15. Let the side AB of rectangle $ABCD$ lies along $4x + 7y + 5 = 0$. As $(-3, 1)$ lies on this line, let it be vertex A . Now, $(1, 1)$ is either vertex C or D .



If $(1, 1)$ is vertex D , then the slope of AD is 0. Hence, AD is not perpendicular to AB . But it is a contradiction as $ABCD$ is a rectangle. Therefore, $(1, 1)$ are the coordinates of vertex C .

CD is a line parallel to AB and passing through C . Therefore, the equation of CD is

$$y - 1 = -\frac{4}{7}(x - 1)$$

or $4x + 7y - 11 = 0$

Also, BC is a line perpendicular to AB and passing through C . Therefore, the equation of BC is

$$y - 1 = \frac{7}{4}(x - 1)$$

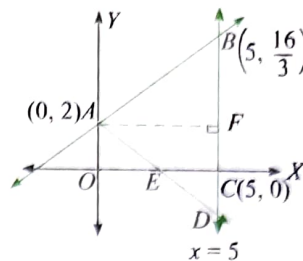
or $7x - 4y - 3 = 0$

Similarly, AD is a line perpendicular to AB and passing through $A(-3, 1)$. Therefore, the equation of line AD is

$$y - 1 = \frac{7}{4}(x + 3)$$

or $7x - 4y + 25 = 0$

16.



$$L_1 = 3y - 2x - 6 = 0$$

The point about which the line is rotated is $A = (0, 2)$. Let the equation of L_2 be $y = mx + 2$. As L_2 will be cutting line $x = 5$ below the x -axis, we have

$$A \equiv (0, 2), B \equiv \left(5, \frac{16}{3}\right), C \equiv (5, 0),$$

$$D \equiv (5, 5m + 2), E \equiv \left(\frac{-2}{m}, 0\right)$$

Now, Area of $AECB$ = Area of $\triangle ADB$ - Area of $\triangle ECD$

$$\begin{aligned} \text{or } \frac{49}{3} &= \frac{1}{2}BD \times AF - \frac{1}{2}EC \times CD \\ &= \frac{1}{2} \left[\frac{16}{3} - (5m + 2) \right] \times 5 - \frac{1}{2} \times \left(5 + \frac{2}{m} \right) [-(5m + 2)] \end{aligned}$$

$$\text{or } \frac{49}{3} = \frac{110m + 12}{6m} \text{ or } m = -1$$

Hence, the equation of L_2 is $x + y = 2$.

17. $m_{AC} = \frac{6-2}{5-3} = 2$

or $m_{BD} = -\frac{1}{2}$

Thus, the equation of BD is

$$(y - 6) = -\frac{1}{2}(x - 5)$$

i.e., $2y + x - 17 = 0$

18. The given line is $x + \sqrt{3}y + 3\sqrt{3} = 0$.

or $y = \left(-\frac{1}{\sqrt{3}}\right)x - 3$

Therefore, the slope of (1) is $-1/\sqrt{3}$.

Let the slope of the required line be m .

Also, the angle between these lines is 60° . Therefore,

$$\tan 60^\circ = \left| \frac{m - (-1/\sqrt{3})}{1 + m(1/\sqrt{3})} \right|$$

or $\sqrt{3} = \left| \frac{\sqrt{3}m + 1}{\sqrt{3} - m} \right|$

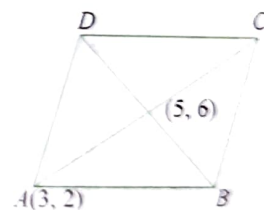
If $\frac{\sqrt{3}m + 1}{\sqrt{3} - m} = \sqrt{3}$

or $m = \frac{1}{\sqrt{3}}$

Using $y = mx + c$, the equation of the required line is

$$y = \frac{1}{\sqrt{3}}x + 0$$

i.e., $x - \sqrt{3}y = 0$ (as the line passes through the origin, $c = 0$)



S.22 Coordinate Geometry

$$\begin{aligned} \text{or } 1 + m^2 &= m^2 + 1 + 2m \\ \text{or } 2m &= 0 \\ \text{or } m &= 0 \end{aligned}$$

Thus, the slope of the required line must be zero, i.e., the line must be parallel to the x -axis.

Exercise 2.2

1. Given lines intersect at $P(2, -1)$.

Slope of line $x + y - 1 = 0$ is -1 .

$$\therefore \tan \theta = -1$$

$$\therefore \cos \theta = -\frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$$

One particle moves 2 units upward from point P on the above line.

Then coordinates of new position obtained by the particle are

$$\left(2 + 2\left(-\frac{1}{\sqrt{2}}\right), -1 + 2 \cdot \frac{1}{\sqrt{2}} \right) \equiv (2 - \sqrt{2}, -1 + \sqrt{2})$$

Slope of line $x - 2y - 4 = 0$ is $1/2$.

$$\therefore \tan \theta = \frac{1}{2}$$

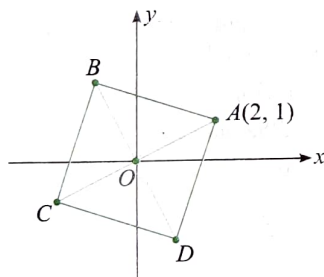
$$\therefore \cos \theta = \frac{2}{\sqrt{5}}, \sin \theta = \frac{1}{\sqrt{5}}$$

Other particle moves 5 units upward from point P on above line.

Then coordinates of new position obtained by the particle are

$$\left(2 + 5\left(\frac{2}{\sqrt{5}}\right), -1 + 5 \cdot \frac{1}{\sqrt{5}} \right) \equiv (2 + 2\sqrt{5}, -1 + \sqrt{5})$$

2.



$ABCD$ is square having centre at origin.

Clearly, $C \equiv (-2, -1)$

$$AO = \sqrt{5}$$

$$\text{Slope of } AO = \frac{1-0}{2-0} = \frac{1}{2}$$

$$\therefore \text{Slope of } BD = -2$$

$$\Rightarrow -2 = \tan \theta$$

$$\therefore \cos \theta = -\frac{1}{\sqrt{5}} \text{ and } \sin \theta = \frac{2}{\sqrt{5}}$$

Since B and D are on BD at a distance $\sqrt{5}$ from O , their coordinates (in some order) will be

$$(0 \pm \sqrt{5} \cos \theta, 0 \pm \sqrt{5} \sin \theta)$$

$$\text{or } (0 \mp 1, 0 \pm 2)$$

$$\text{or } (-1, 2) \text{ and } (1, -2)$$

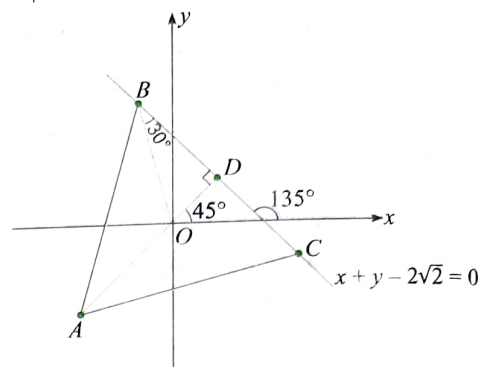
3. Line is passing through the point $P(x_1, y_1)$ and making an angle α with x -axis.

One of the points on this line at distance r from the point P is given by $(x_1 + r \cos \alpha, y_1 + r \sin \alpha)$.

If this point is Q , then it lies on the line. So, $A(x_1 + r \cos \alpha) + B(y_1 + r \sin \alpha) + C = 0$.

$$\therefore r = \frac{|Ax_1 + By_1 + C|}{|A \cos \alpha + B \sin \alpha|}$$

4.



As shown in the figure, equilateral triangle ABC is such that B and C lie on the given line $x + y - 2\sqrt{2} = 0$.

AD is perpendicular to BC , which passes through centroid.

$$OD = \frac{|0 + 0 - 2\sqrt{2}|}{\sqrt{2}} = 2$$

$\therefore OA = 4$ (As centroid divides median in the ratio $2:1$)

Slope of AD is 1 , i.e., $\tan 45^\circ$, where 45° is the inclination of AD with x -axis.

$$\therefore A \equiv (0 - 4 \cos 45^\circ, 0 - 4 \sin 45^\circ) \equiv (-2\sqrt{2}, -2\sqrt{2})$$

$$\text{and } D \equiv (0 + 2 \cos 45^\circ, 0 + 2 \sin 45^\circ) \equiv (\sqrt{2}, \sqrt{2})$$

$$\text{Now, } DC = DB = 2 \cot 30^\circ = 2\sqrt{3}$$

$$\begin{aligned} \text{Therefore, } B &\equiv (\sqrt{2} + 2\sqrt{3} \cos 135^\circ, \sqrt{2} + 2\sqrt{3} \sin 135^\circ) \\ &\equiv (\sqrt{2} - \sqrt{6}, \sqrt{2} + \sqrt{6}) \end{aligned}$$

$$\begin{aligned} \text{And } C &\equiv (\sqrt{2} - 2\sqrt{3} \cos 135^\circ, \sqrt{2} - 2\sqrt{3} \sin 135^\circ) \\ &\equiv (\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6}) \end{aligned}$$

Exercise 2.3

1. Let the required point be $P(0, \alpha)$. It is given that the length of the perpendicular from $P(0, \alpha)$ on $4x - 3y - 12 = 0$ is 3 . Therefore,

$$\left| \frac{4(0) - 3\alpha - 12}{\sqrt{4^2 + (-3)^2}} \right| = 3$$

$$\text{or } |3\alpha + 12| = 15$$

$$\text{or } |\alpha + 4| = 5$$

$$\text{or } \alpha + 4 = \pm 5$$

$$\text{or } \alpha = 1, -9$$

Hence, the required points are $(0, 1)$ and $(0, -9)$.

$$2. \text{ Here, } p = \left| \frac{-k}{\sqrt{\sec^2 \alpha + \csc^2 \alpha}} \right|, p' = \left| \frac{-k \cos 2\alpha}{\sqrt{\cos^2 \alpha + \sin^2 \alpha}} \right|$$

$$\text{Hence, } 4p^2 + p'^2 = \frac{4k^2}{\sec^2 \alpha + \csc^2 \alpha} + \frac{k^2(\cos^2 \alpha - \sin^2 \alpha)^2}{1}$$

$$= 4k^2 \sin^2 \alpha \cos^2 \alpha + k^2(\cos^4 \alpha + \sin^4 \alpha) - 2k^2 \cos^2 \alpha \times \sin^2 \alpha$$

$$= k^2(\sin^2 \alpha + \cos^2 \alpha)^2 = k^2$$

$$3. p_1 = \frac{m^2 + 2m + 1}{\sqrt{2}} = \frac{(m+1)^2}{\sqrt{2}}$$

$$p_2 = \frac{|mm' + m + m' + 1|}{\sqrt{2}} = \frac{|(m'+1)(m+1)|}{\sqrt{2}}$$

$$p_3 = \frac{(m')^2 + 2m' + 1}{\sqrt{2}} = \frac{(m' + 1)^2}{\sqrt{2}}$$

Clearly, p_1, p_2 , and p_3 are in GP.

4. Lines $3x + 4y + 2 = 0$ and $3x + 4y + 5 = 0$ are on the same side of the origin.

The distance between these lines is

$$d_1 = \left| \frac{2-5}{\sqrt{3^2+4^2}} \right| = \frac{3}{5}$$

Lines $3x + 4y + 2 = 0$ and $3x + 4y - 5 = 0$ are on the opposite sides of the origin.

The distance between these lines is

$$d_2 = \left| \frac{2+5}{\sqrt{3^2+4^2}} \right| = \frac{7}{5}$$

Thus, $3x + 4y + 2 = 0$ divides the distance between $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ in the ratio $d_1 : d_2$, i.e., 3 : 7.

5. Incentre of the triangle is the point which is equidistant from all the sides of triangle.

Clearly, distance of origin from all the three given lines is π .

Also, (0, 0) is interior point of triangle which can be verified by drawing lines on the plane.

So, incentre of the triangle is (0, 0).

6. The equation of any line parallel to $3x - 4y - 5 = 0$ is

$$3x - 4y + \lambda = 0$$

Distance of (1) from $3x - 4y - 5 = 0$ is 1 unit. Therefore,

$$\frac{|5 + \lambda|}{5} = 1$$

i.e., $\lambda = -10$ or 0

Therefore, the required line is $3x - 4y - 10 = 0$ or $3x - 4y = 0$. These are the equations of the required lines.

7. Since the perpendicular distance between the given lines is $\sqrt{2}$, the required line is a straight line perpendicular to the given parallel lines and passes through $(-5, 4)$.

Any line perpendicular to the given lines is

$$x - y + k = 0$$

This line passes through $(-5, 4)$. Therefore,

$$-5 - 4 + k = 0$$

or $k = 9$

Hence, the required line is

$$x - y + 9 = 0$$

Exercise 2.4

1. Given lines are parallel lines.

$$\text{Let } L_1(x, y) = 2x + 3y - 4$$

$$\text{and } L_2(x, y) = 6x + 9y + 8$$

$$L_1(8, -9) = 2(8) + 3(-9) + 4 = -7 < 0$$

$$L_2(8, -9) = 6(8) + 9(-9) + 8 = -25 < 0$$

Hence, both the lines lie on the same side of point (8, -9).

2. We have $L(x, y) = 3x - 8y - 7$

$$(i) L(-3, -4) = 3(-3) - 8(-4) - 7 = 16 > 0$$

$$L(1, 2) = 3(1) - 8(2) - 7 = -20 < 0$$

So, points are on opposite sides of the line.

$$(ii) L(-1, -1) = 3(-1) - 8(-1) - 7 = -2 < 0$$

$$L(3, 7) = 3(3) - 8(7) - 7 = -54 < 0$$

So, points are on the same side of the line.

3. We have $L(x, y) = 2x + 3y - 6$

$$\therefore L(\alpha, 2 + \alpha) = 5\alpha$$

$$\text{and } L\left(\frac{3}{2}\alpha, \alpha^2\right) = 3\alpha + 3\alpha^2 - 6$$

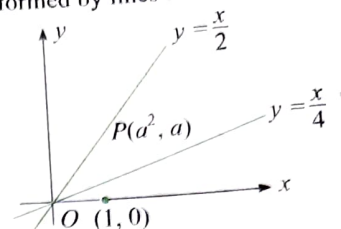
Given points lie on the opposite sides of the line if

$$5\alpha(3\alpha + 3\alpha^2 - 6) < 0$$

$$\Rightarrow \alpha(\alpha + 2)(\alpha - 1) < 0$$

$$\Rightarrow \alpha \in (-\infty, -2) \cup (0, 1)$$

4. Acute angle is formed by lines in first and third quadrants.



But $a^2 > 0$, and hence, point $P(a^2, a)$ lies in the first quadrant.

We have $a - \frac{a^2}{4} > 0$ and $a - \frac{a^2}{2} < 0$

$$\Rightarrow a \in (0, 4) \text{ and } a \in (-\infty, 0) \cup (2, \infty)$$

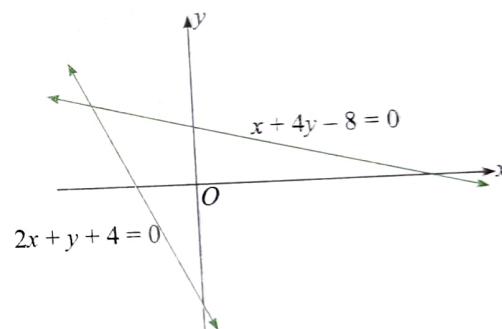
$$\Rightarrow a \in (2, 4)$$

5. We have lines

$$2x + y + 4 = 0$$

$$\text{and } x + 4y - 8 = 0$$

Lines are drawn as shown in the following figure.



$$\text{Let } L_1(x, y) = 2x + y + 4$$

$$\text{and } L_2(x, y) = x + 4y - 8$$

$$L_1(0, 0) = 4 > 0$$

So, $(a, 3a)$ lies above the first line if

$$L_1(a, 3a) = 2a + 3a + 4 > 0$$

$$\text{or } a > -4/5$$

$$L_2(0, 0) = -8 < 0$$

So, $(a, 3a)$ lies below the second line if

$$L_2(a, 3a) = a + 12a - 8 < 0$$

$$\text{or } a < 8/13$$

From (3) and (4),

$$-4/5 < a < 8/13$$

6. Given lines are

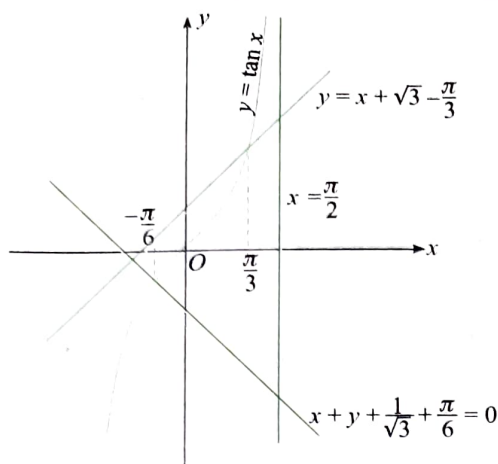
$$y = x + \sqrt{3} - \frac{\pi}{3}$$

$$x + y + \frac{1}{\sqrt{3}} + \frac{\pi}{6} = 0$$

$$\text{and } x - \frac{\pi}{2} = 0$$

Variable point $(\alpha, \tan \alpha)$ lies on the curve $y = \tan x$.

So, we draw the given lines and curve $y = \tan x$.



By hit and trial method, we find that line (1) cuts the curve

$y = \tan x$ for $x = \frac{\pi}{3}$ and line (2) cuts the curve for $x = -\frac{\pi}{6}$.

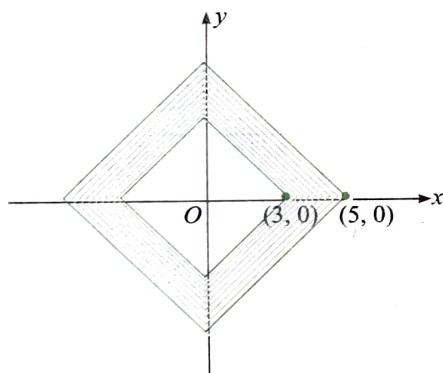
Hence, point $(\alpha, \tan \alpha)$ lies inside the triangle if $-\frac{\pi}{6} < \alpha < \frac{\pi}{3}$.

7. We have $3 \leq |x| + |y| \leq 5$.

$|x| + |y| \leq 5$ means that points lie inside the square endpoints of whose diagonal are $(\pm 5, 0)$ and $(0, \pm 5)$.

$|x| + |y| \geq 3$ means that points lie outside the square endpoints of whose diagonal are $(\pm 3, 0)$ and $(0, \pm 3)$.

So, common region is as shown in the figure.

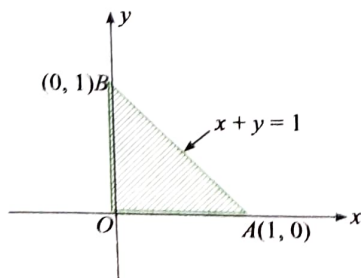


$$\text{Area of region} = 4 \left(\frac{1}{2} \times 5 \times 5 - \frac{1}{2} \times 3 \times 3 \right) = 32 \text{ sq. units}$$

8. If $x, y > 0$, then we have

$$x + y + x + y \leq 2 \quad \text{or} \quad x + y \leq 1$$

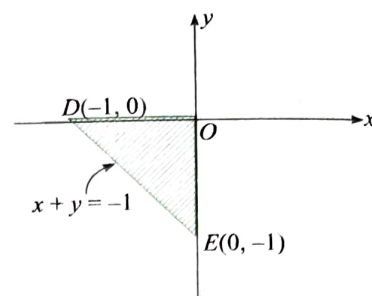
These inequalities form a triangular region in the first quadrant formed by axes and line $x + y = 1$.



If $x, y < 0$, then we have

$$-x - y - x - y \leq 2 \quad \text{or} \quad x + y \geq -1$$

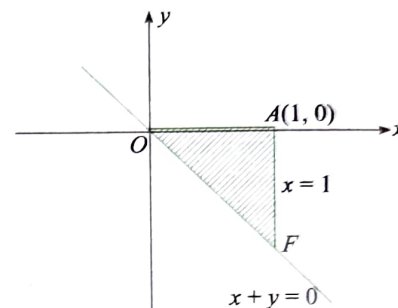
These inequalities form a triangular region in the third quadrant formed by y -axis, $x + y = 0$ and line $x + y = -1$.



If $x > 0, y < 0$ and $x + y > 0$, then we have

$$x - y + x + y \leq 2 \quad \text{or} \quad x \leq 1$$

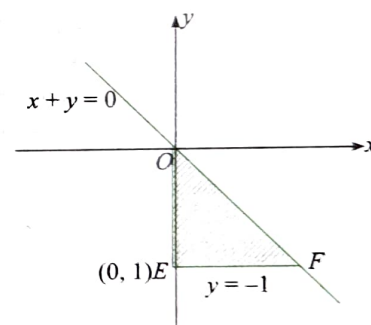
These inequalities form a triangular region in the fourth quadrant formed by x -axis, $x + y = 0$ and $x = 1$.



If $x > 0, y < 0$ and $x + y < 0$, then we have

$$x - y - x - y \leq 2 \quad \text{or} \quad y \geq -1$$

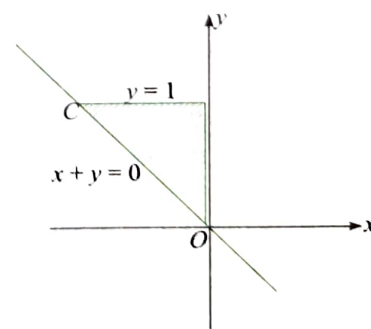
These inequalities form a triangular region in the fourth quadrant formed by y -axis, $x + y = 0$ and $y = -1$.



If $x < 0, y > 0$ and $x + y > 0$, then we have

$$-x + y + x + y \leq 2 \quad \text{or} \quad y \leq 1$$

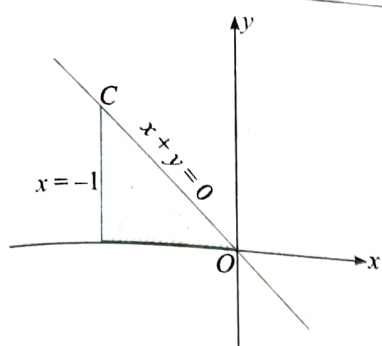
These inequalities form a triangular region in the second quadrant formed by y -axis, $x + y = 0$ and $y = 1$.



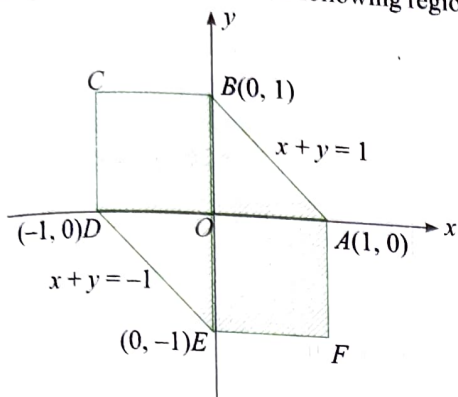
If $x < 0, y > 0$ and $x + y < 0$, then we have

$$-x + y - x - y \leq 2 \quad \text{or} \quad x \geq -1$$

These inequalities form a triangular region in the second quadrant formed by x -axis, $x + y = 0$ and $x = -1$.



Combining all the regions, we have following region in the plane



Area of the above region = $3 \times$ Area of square having side length 1 unit.
= 3 sq. unit

Exercise 2.5

1. Firstly, make the constant terms (c_1, c_2) positive.

$$3x - 4y + 7 = 0$$

$$\text{and } -12x - 5y + 2 = 0$$

$$\therefore a_1 a_2 + b_1 b_2 = (3)(-12) + (-4)(-5) \\ = -36 + 20 = -16$$

Hence, “-” sign gives the obtuse bisector.

Therefore, the obtuse bisector is

$$\frac{(3x - 4y + 7)}{\sqrt{(3)^2 + (-4)^2}} = - \frac{(-12x - 5y + 2)}{\sqrt{(-12)^2 + (-5)^2}}$$

$$\text{or } 13(3x - 4y + 7) = -5(-12x - 5y + 2)$$

$$\text{or } 21x + 77y - 101 = 0$$

2. Mirrors are always along the bisectors of incident ray and reflected ray.

Now, the equations of bisectors are

$$\frac{3x - 4y - 3}{5} = \pm \frac{24x + 7y + 5}{25}$$

Therefore, the equations of mirror are

$$9x + 27y + 20 = 0$$

$$\text{or } 39x - 13y - 10 = 0$$

3. The diagonals of rhombus are along the angle bisectors of sides.

The bisector of the longer diagonal is that of acute angle bisector.

The given sides of rhombus are

$$x + 2y + 2 = 0 \quad (1)$$

$$\text{and } -2x - y + 3 = 0 \quad (2)$$

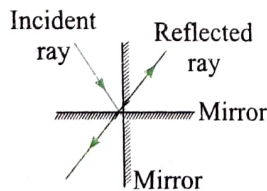
$$\text{Now, } a_1 a_2 + b_1 b_2 = -2 - 2 = -4 < 0$$

The acute angle bisector is

$$\frac{x + 2y + 2}{5} = \frac{-2x - y + 3}{5}$$

$$\text{or } 3x + 3y - 1 = 0$$

Therefore, the slope of the longer diagonal is -1.



4. B and C will be the images of A on $y + x = 0$ and $y - x = 0$, respectively. Thus,

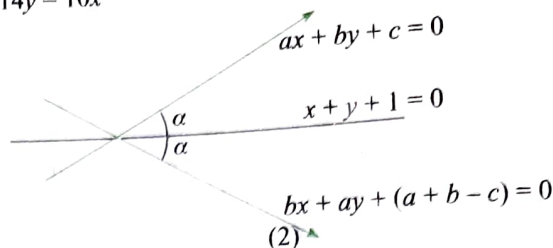
$$B \equiv (-7, -5), C \equiv (7, 5)$$

Hence, the equation of BC is

$$y - 5 = \frac{-5 - 5}{-7 - 7}(x - 7)$$

$$\text{i.e., } 14y = 10x$$

5.



We have lines

$$ax + by + c = 0 \quad (1)$$

$$x + y + 1 = 0 \quad (2)$$

$$bx + ay + (a + b - c) = 0 \quad (3)$$

Equations of angle bisectors of (1) and (3) are

$$\frac{|ax + by + c|}{\sqrt{a^2 + b^2}} = \frac{|bx + ay + a + b - c|}{\sqrt{b^2 + a^2}}$$

$$\text{or } ax + by + c = \pm(bx + ay + a + b - c)$$

$$\text{or } x + y + 1 = 0 \text{ and } (a - b)x + (b - a)y = a + b - 2c$$

Since $x + y + 1 = 0$ is one of the bisectors of lines (1) and (3), reflection of line (1) in line (2) is line (3).

6. Clearly, the third altitude will be the bisector of obtuse angle between the given altitudes.

Given two altitudes are:

$$\sqrt{3}x - y + 8 - 4\sqrt{3} = 0 \quad (1)$$

$$\text{and } -\sqrt{3}x - y + 12 + 4\sqrt{3} = 0 \quad (2)$$

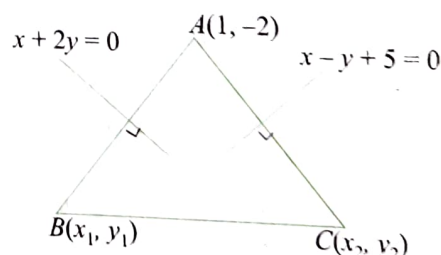
$$\text{We have } a_1 a_2 + b_1 b_2 = -\sqrt{3}\sqrt{3} + (-1)(-1) = -2 < 0$$

Therefore, bisector of obtuse angle is

$$\frac{\sqrt{3}x - y + 8 - 4\sqrt{3}}{\sqrt{3} + 1} = - \frac{-\sqrt{3}x - y + 12 + 4\sqrt{3}}{\sqrt{3} + 1}$$

$$\text{or } y = 10$$

7.



Since the given lines are perpendicular bisectors of the sides as shown in the figure, points B and C are the images of the point A in these lines. Therefore,

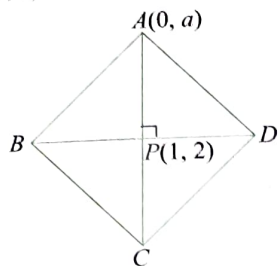
$$\frac{x_1 - 1}{1} = \frac{y_1 + 2}{2} = - \frac{2(1 - 4)}{1 + 4}$$

$$\text{and } \frac{x_2 - 1}{1} = \frac{y_2 + 2}{-1} = - \frac{2(1 + 2 + 5)}{1 + 1}$$

$$\therefore B(x_1, y_1) \equiv (11/5, 2/5) \text{ and } C(x_2, y_2) \equiv (-7, 6)$$

Hence, the line passing through the points B and C is $14x + 23y - 40 = 0$.

8. A being on the y -axis may be chosen as $(0, a)$. The diagonals intersect at $P(1, 2)$.



Again, we know that the diagonals will be parallel to the bisectors of the two given lines $y = x + 2$ and $y = 7x + 3$, i.e.,

$$\frac{x - y + 2}{\sqrt{2}} = \pm \frac{7x - y + 3}{5\sqrt{2}}$$

or $5x - 5y + 10 = \pm(7x - y + 3)$

or $2x + 4y - 7 = 0$ and $12x - 6y + 13 = 0$

$\therefore$ Slope of diagonals are $m_1 = -1/2$ and $m_2 = 2$. Let diagonal d_1 be parallel to $2x + 4y - 7 = 0$ and diagonal d_2 be parallel to $12x - 6y + 13 = 0$. The vertex A could be on any of the two diagonals. Hence, the slope of AP is either $-1/2$ or 2 . Therefore,

$$\frac{2 - a}{1 - 0} = 2 \text{ or } -\frac{1}{2}$$

i.e., $a = 0$ or $\frac{5}{2}$

Hence, A is $(0, 0)$ or $(0, 5/2)$.

Exercise 2.6

1. $(a - 2b)x + (a + 3b)y + 3a + 4b = 0$

or $a(x + y + 3) + b(-2x + 3y + 4) = 0$

which represents a family of straight lines passing through the point of intersection of $x + y + 3 = 0$ and $-2x + 3y + 4 = 0$, i.e., $(-1, -2)$.

2. a, b, c are in HP. Then,

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \quad (1)$$

The given line is

$$\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0 \quad (2)$$

Subtracting (1) from (2), we get

$$\frac{1}{a}(x - 1) + \frac{1}{b}(y + 2) = 0$$

Since $a \neq 0$ and $b \neq 0$, we get

$$x - 1 = 0 \text{ and } y + 2 = 0$$

or $x = 1$ and $y = -2$

3. Let the equation of the variable line be

$$ax + by + c = 0$$

Then according to the given condition, we get

$$\frac{2a + c}{\sqrt{a^2 + b^2}} + \frac{2b + c}{\sqrt{a^2 + b^2}} + \frac{-2a - 2b + c}{\sqrt{a^2 + b^2}} = 0$$

or $c = 0$

which shows that the line passes through $(0, 0)$ for all values of a and b .

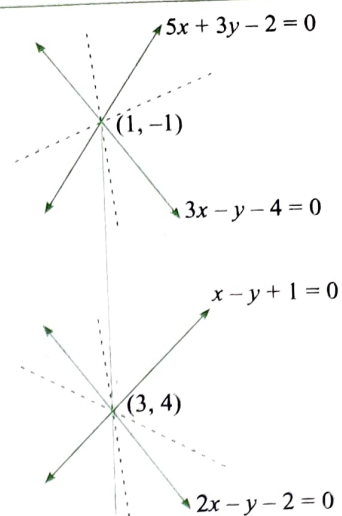
4. Lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ are concurrent at $(1, -1)$ and

lines $x - y + 1 + \lambda_2(2x - y - 2) = 0$ are concurrent at $(3, 4)$.

Thus, the equation of line common to both families is

$$y - 4 = \frac{-1 - 4}{1 - 3}(x - 3)$$

i.e., $5x - 2y - 7 = 0$



5.
$$\begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & 1 \\ a & b & -c \end{vmatrix} = 0$$

or $a + 5b - 3c = 0$

or $-\frac{a}{3} - \frac{5}{3}b + c = 0$

Hence, $2ax + 3by + c = 0$ is concurrent at $2x = -1/3$ and $3y = -5/3$.

So, $x = -1/6, y = -5/9$.

Exercises

Single Correct Answer Type

1. (1) The coordinates of the vertices of the square are $A(0, 0)$, $B(0, 1)$, $C(1, 1)$, and $D(1, 0)$.

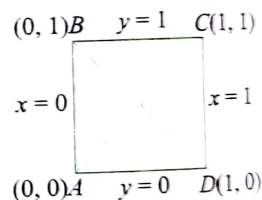
Now, the equation of AC is

$$y = x$$

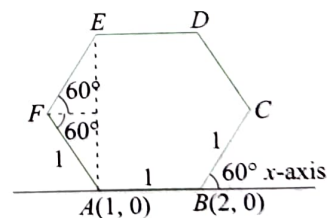
and that of BD is

$$y - 1 = -\frac{1}{1}(x - 0)$$

or $x + y = 1$



2. (3)



$$AB = 1$$

or $C \equiv (2 + 1 \times \cos 60^\circ, 1 \times \sin 60^\circ) \equiv \left(\frac{5}{2}, \frac{\sqrt{3}}{2}\right)$

$$E \equiv (1, 1 \times \sin 60^\circ + 1 \times \sin 60^\circ) \equiv (1, \sqrt{3})$$

Therefore, the equation of CE is

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x - 1)$$

3. (2) The equation of the line joining the points $(2, -1)$ and $(5, -3)$ is given by

$$y + 1 = \frac{-1 + 3}{2 - 5}(x - 2)$$

or $2x + 3y - 1 = 0$

Since $(x_1, 4)$ and $(-2, y_1)$ lie on $2x + 3y - 1 = 0$, we have

$$2x_1 + 12 - 1 = 0 \text{ or } x_1 = -\frac{11}{2}$$

and $-4 + 3y_1 - 1 = 0$ or $y_1 = \frac{5}{3}$

Thus, (x_1, y_1) satisfies $2x + 6y + 1 = 0$.

4. (1) The line perpendicular to $x \sec \theta + y \operatorname{cosec} \theta = a$ is $x \operatorname{cosec} \theta - y \sec \theta = \lambda$

This line passes through the point $(a \cos^3 \theta, a \sin^3 \theta)$. Then, $(a \cos^3 \theta) \operatorname{cosec} \theta - (a \sin^3 \theta) \sec \theta = \lambda$

$$\text{or } \lambda = a \left(\frac{\cos^3 \theta}{\sin \theta} - \frac{\sin^3 \theta}{\cos \theta} \right)$$

$$= a \left(\frac{\cos 2\theta}{\cos \theta \sin \theta} \right)$$

5. (3) Clearly, the equation of PQ in the new position is $x = 2$.

6. (3) Let the line intersect the axes at A and B .

Let midpoint of AB be $P(h, k)$.

$$\therefore A(2h, 0) \text{ and } B(0, 2k)$$

So, equation of line in intercept form is

$$\frac{x}{2h} + \frac{y}{2k} = 1$$

Sum of intercepts is constant c .

$$\therefore 2h + 2k = c$$

$$\therefore 2x + 2y = c$$

This is the required locus of point P .

7. (4) Given equation of line is $y = mx + 2$

$$\Rightarrow \frac{x}{-\frac{2}{m}} + \frac{y}{2} = 1$$

It is given that x -intercept of the above line is greater than $\frac{1}{2}$.

$$\therefore -\frac{2}{m} > \frac{1}{2}$$

$$\Rightarrow -\frac{2}{m} - \frac{1}{2} > 0$$

$$\Rightarrow -\left(\frac{4+m}{2m}\right) > 0$$

$$\Rightarrow \frac{4+m}{2m} < 0$$

$$\Rightarrow -4 < m < 0$$

8. (1) Given that the slope is $-\sqrt{3}$.

Therefore, the line is

$$y = -\sqrt{3}x + c$$

$$\text{or } \sqrt{3}x + y = c$$

$$\text{Now, } \left| \frac{c}{2} \right| = 4 \text{ or } c = \pm 8$$

$$\text{or } x\sqrt{3} + y = \pm 8$$

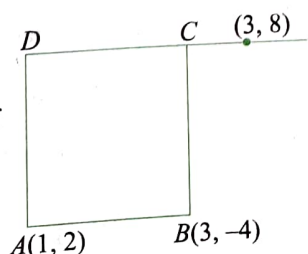
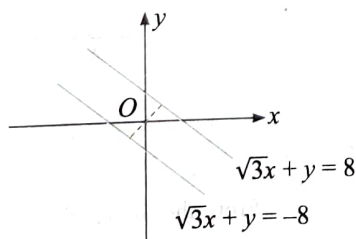
9. (4) $m_{AB} = \frac{-4-2}{3-1} = -3$

Thus, the equation of CD is $y - 8 = -3(x - 3)$, i.e., $y + 3x = 17$. The equation of the right bisector of AB is

$$y + 1 = \frac{1}{3}(x - 2)$$

$$\text{or } 3y = x - 5$$

Solving it with line CD , we get $x = 28/5$, $y = 1/5$. Thus, the midpoint of CD is $(28/5, 1/5)$.

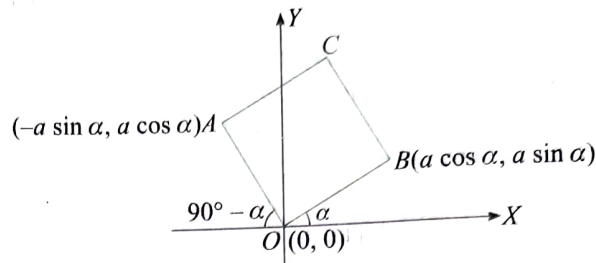


10. (1) Let the line cuts the axes at points $A(a, 0)$ and $B(0, b)$. Now, given that $(-4, 3)$ divides AB in the ratio $5 : 3$. Then, $-4 = 3a/8$ and $3 = 5b/8$. Therefore, $a = -32/3$ and $b = 24/5$. Then using the intercept form $x/a + y/b = 1$, the equation of line is

$$-\frac{3x}{32} + \frac{5y}{24} = 1$$

$$\text{or } 9x - 20y + 96 = 0$$

11. (3) The coordinates of A and B are as shown in the figure.



The equation of the diagonal AB is

$$y - a \sin \alpha = \frac{a \cos \alpha - a \sin \alpha}{-a \sin \alpha - a \cos \alpha} (x - a \cos \alpha)$$

$$\text{or } y (\cos \alpha + \sin \alpha) - a (\sin \alpha \cos \alpha + \sin^2 \alpha)$$

$$= -(\cos \alpha - \sin \alpha) x + a \cos \alpha (\cos \alpha - \sin \alpha)$$

$$\text{or } y (\cos \alpha + \sin \alpha) + x (\cos \alpha - \sin \alpha) = a$$

12. (3) The slope of QR is

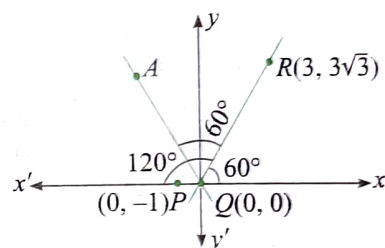
$$(3\sqrt{3} - 0)/(3 - 0) = \sqrt{3}$$

$$\text{i.e., } \theta = 60^\circ$$

Clearly, $\angle PQR = 120^\circ$.

AQ is the angle bisector of the angle. So, line AQ makes 120° with the positive direction of the x -axis. Therefore, the equation of the bisector of $\angle PQR$ is

$$y = x \tan 120^\circ \text{ or } y = -\sqrt{3}x, \text{ i.e., } \sqrt{3}x + y = 0.$$



13. (1) Obviously, the line through Q is at the greatest distance from point P when it is perpendicular to PQ . Now, the slope of line PQ is $m_{PQ} = -1/2$. Then the slope of perpendicular line passing through Q is

$$y - 2 = 2(x - 1)$$

$$\text{or } 2x - y = 0$$

14. (3) The slope of BD is $8/15$ and the angle made by BD with DC and BC is 45° . So, let the slope of DC be m . Then,

$$\tan 45^\circ = \pm \frac{m - \frac{8}{15}}{1 + \frac{8}{15}m}$$

$$\text{or } (15 + 8m) = \pm(15m - 8)$$

$$\text{or } m = \frac{23}{7} \text{ and } -\frac{7}{23}$$

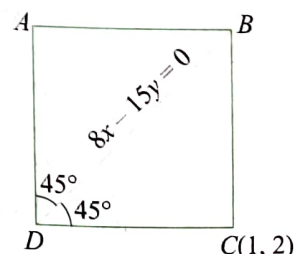
Hence, the equations of DC and BC are

$$y - 2 = \frac{23}{7}(x - 1)$$

$$\text{or } 23x - 7y - 9 = 0$$

$$\text{and } y - 2 = -\frac{7}{23}(x - 1)$$

$$\text{or } 7x + 23y - 53 = 0$$



15. (2) Clearly, distance between the parallel lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{a} + \frac{y}{b} = 2$ is same as the distance between the parallel lines $\frac{x}{b} + \frac{y}{a} = 1$ and $\frac{x}{b} + \frac{y}{a} = 2$.

Therefore, quadrilateral is rhombus.

16. (2) Equation of line through point $A(6, 0)$ and having slope m is
 $(y - 0) = m(x - 6), m > 0$
 or $mx - y - 6m = 0, m > 0$... (1)

Distance of this line from point $B(1, 3)$ is 5.

$$\text{So, } \left| \frac{m - 3 - 6m}{\sqrt{1 + m^2}} \right| = 5$$

$$\Rightarrow (3 + 5m)^2 = 25(1 + m^2)$$

$$\Rightarrow 9 + 30m = 25$$

$$\Rightarrow m = \frac{16}{30} = \frac{8}{15}$$

17. (3) The family of line through the given lines is

$$L \equiv x - 7y + 5 + \lambda(x + 3y - 2) = 0 \quad (1)$$

For line $L = 0$ in the diagram, the distance of any point say $(2, 0)$ on the line $x + 3y - 2 = 0$ from the line $x - 7y + 5 = 0$ and the line $L = 0$ must be the same. Therefore,

$$\left| \frac{2 + 5}{\sqrt{50}} \right| = \left| \frac{2 + 2\lambda + 5 - 2\lambda}{\sqrt{(1 + \lambda)^2 + (3\lambda - 7)^2}} \right|$$

$$\text{or } 10\lambda^2 - 40\lambda = 0$$

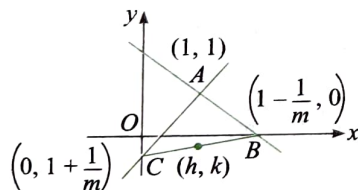
$$\text{i.e., } \lambda = 4 \text{ or } 0$$

$$\text{Hence, } L = 0, \lambda = 4.$$

Therefore, the required line is $5x + 5y - 3 = 0$.

18. (1) The equation of line AB is $y - 1 = m(x - 1)$. Therefore, the equation of line AC is

$$y - 1 = -\frac{1}{m}(x - 1)$$



$$2h = 1 - \frac{1}{m}$$

$$2k = 1 + \frac{1}{m}$$

Eliminating m , we have locus $x + y = 1$.

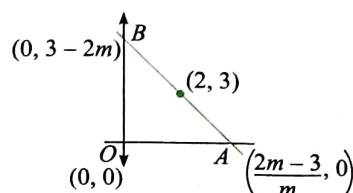
19. (3) The equation of any line through $(2, 3)$ is

$$y - 3 = m(x - 2)$$

$$\text{or } y = mx - 2m + 3$$

From the figure, the area of $\triangle OAB$ is ± 12 , that is,

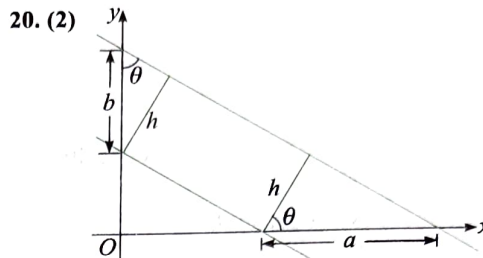
$$\frac{1}{2} \left(\frac{2m - 3}{m} \right) (3 - 2m) = \pm 12$$



Taking positive sign, we get $(2m + 3)^2 = 0$. This gives one value of m as $-3/2$. Taking negative sign, we get

$$4m^2 - 36m + 9 = 0 \quad (D > 0)$$

This is a quadratic in m which gives two values of m . Hence, three straight lines are possible.



From the figure,

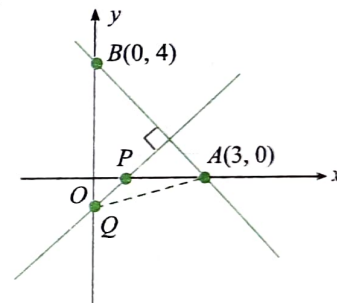
$$\cos \theta = \frac{h}{a} \text{ and } \sin \theta = \frac{h}{b}$$

Squaring these and then adding, we get

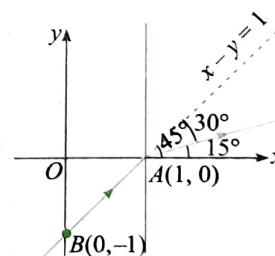
$$1 = \frac{h^2}{a^2} + \frac{h^2}{b^2}$$

$$\therefore h = \frac{ab}{\sqrt{a^2 + b^2}}$$

21. (3) Clearly, the circumcenter of triangle ABQ will lie on the perpendicular bisector of line AB . Now, the equation of perpendicular bisector of line AB is $3x - 4y + 7/2 = 0$. Hence, the locus of circumcenter is $6x - 8y + 7 = 0$.



22. (4)



From the figure, the refracted ray makes an angle of 15° with the positive direction of the x -axis and passes through the point $(1, 0)$. Its equation is

$$(y - 0) = (\tan 15^\circ)(x - 1)$$

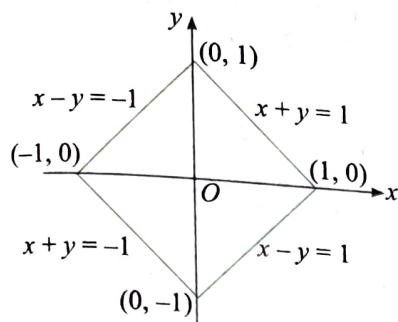
$$\text{or } y = (2 - \sqrt{3})(x - 1)$$

23. (1) Solving $3x + 4y = 9, y = mx + 1$, we get

$$x = \frac{5}{3 + 4m}$$

Here, x is an integer if $3 + 4m = 1, -1, 5, -5$. Hence, $m = -2/4, -4/4, 2/4, -8/4$. So, m has two integral values.

14. (1) Let the two perpendicular lines be taken as the coordinate axes. If (h, k) is any point on the locus, then according to the given condition, $|h| + |k| = 1$. Hence, the locus of (h, k) is $|x| + |y| = 1$. This consists of four line segments enclosing a square as shown in the figure.



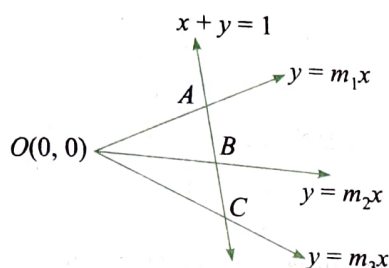
15. (3) If normal to the line makes an angle α with x -axis, then equation of line is

$x \cos \alpha + y \sin \alpha = p$, where p is distance of line from origin.

Here, $p = 2$

So, equation of lines is $x \cos \alpha + y \sin \alpha = 2$.

16. (1)



Solving the equations of the lines, we get

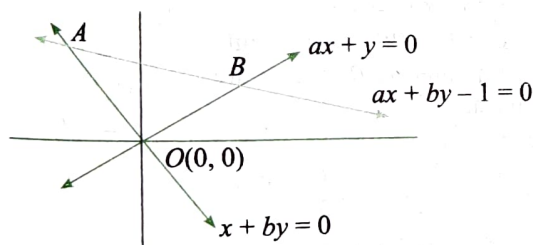
$$A \equiv \left(\frac{1}{1+m_1}, \frac{m_1}{1+m_1} \right), C \equiv \left(\frac{1}{1+m_3}, \frac{m_3}{1+m_3} \right)$$

If $AB = BC$, then the midpoint of AC lies on

$$y = m_2x$$

$$\text{or } \frac{\frac{m_1}{1+m_1} + \frac{m_3}{1+m_3}}{2} = m_2 \left[\frac{\frac{1}{1+m_1} + \frac{1}{1+m_3}}{2} \right]$$

17. (2) Lines $ax + y = 0$ and $x + by = 0$ intersect at $O(0, 0)$.



Hence, if AB subtends right angle at $O(0, 0)$, then $ax + y = 0$ and $x + by = 0$ are perpendicular to each other. So,

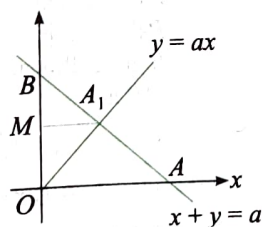
$$(-a)\left(-\frac{1}{b}\right) = -1$$

$$\text{or } a + b = 0$$

28. (4) $B \equiv (0, a)$, $A_1 \equiv \left(\frac{a}{1+a}, \frac{a^2}{1+a} \right)$

$$\Delta_{OA_1B} = \frac{1}{2}(OB) A_1M$$

$$= \frac{1}{2} |a| \left| \frac{a}{1+a} \right| = \frac{1}{2} \frac{a^2}{|1+a|}$$



29. (3) Given $\ar \Delta AOC = 2(\ar \Delta BOC)$

$$\text{or } \frac{1}{2}(OA)(x_1) = \frac{2 \times 1}{2}(OB)(x_1)$$

$$\text{or } a = 2b$$

The equation of AB is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad (1)$$

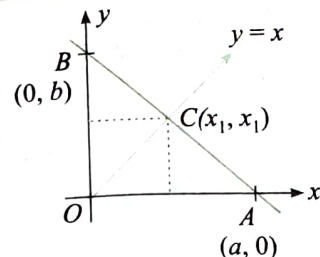
$$\text{or } \frac{x}{2b} + \frac{y}{b} = 1 \quad (2)$$

Since point C lies on line (2), we have

$$\frac{x_1}{2b} + \frac{x_1}{b} = 1$$

$$\text{or } x_1 = \frac{2b}{3} = \frac{a}{3}$$

$$\text{or } C \equiv \left(\frac{2b}{3}, \frac{2b}{3} \right)$$



30. (2) The coordinates of A are $(0, 4)$ and those of B are $(3, 0)$.

$$BC = AB = \sqrt{3^2 + 4^2} = 5$$

$$\text{or } BL = BC \sin \theta \text{ and } CL = BC \cos \theta$$

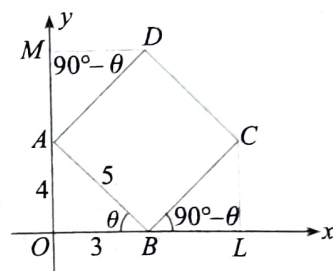
$$\text{or } BL = 5 \times \frac{4}{5} = 4 \text{ and}$$

$$CL = 5 \times \frac{3}{5} = 3$$

Similarly, $MD = 4$ and $AM = 3$.

So, the coordinates of C are $(OB + BL, CL) \equiv (7, 3)$ and those of D are $(MD, OA + AM) \equiv (4, 7)$.

The coordinates of the vertex farthest from the origin are $(4, 7)$.



31. (2) The given lines are

$$ax \pm by \pm c = 0$$

$$\text{or } \frac{x}{\pm c/a} + \frac{y}{\pm c/b} = 1$$

The vertices of rhombus are at $A(c/a, 0)$, $C(-c/a, 0)$, $B(0, c/b)$, $D(0, -c/b)$. Therefore, the diagonals AC and BD of quadrilateral $ABCD$ are perpendicular. Hence, it is a rhombus whose area is given by

$$\frac{1}{2} \times AC \times BD = \frac{1}{2} \times \frac{2c}{a} \times \frac{2c}{b} = \frac{2c^2}{ab}$$

32. (4) Let $ABCD$ be square having vertex $A(-1, 1)$.

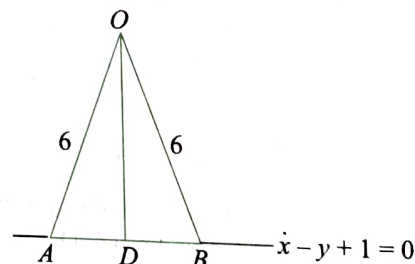
One diagonal is $3x - 4y + 8 = 0$; clearly, it is BD . Let O be centre of the square.

$$\therefore OA = OB = \left| \frac{-3 - 4 + 8}{5} \right| = \frac{1}{5}$$

Area of square $= 4 \times \Delta AOB$

$$= 4 \left(\frac{1}{2} \times OA \times OB \right) = 4 \left(\frac{1}{2} \times \frac{1}{5} \times \frac{1}{5} \right) = \frac{2}{25} \text{ sq. unit}$$

33. (4)



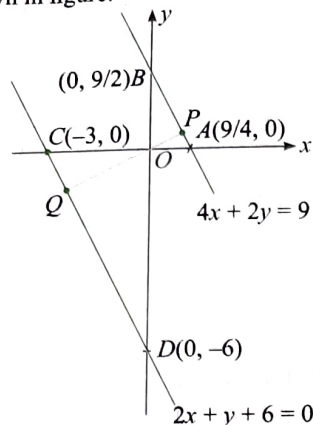
Let D be the midpoint of AB .

$$OD = \frac{1}{\sqrt{2}}$$

$$AD = \sqrt{6^2 - OD^2} = \sqrt{36 - \frac{1}{2}} = \sqrt{\frac{71}{2}}$$

$$\therefore \text{Area of triangle} = \frac{1}{2} \times AB \times OD = \sqrt{\frac{71}{2}} \times \sqrt{\frac{1}{2}} = \frac{\sqrt{71}}{2}$$

34. (2) Let any line through the origin meets the given lines at P and Q as shown in figure.



Now, from the figure, triangles OAP and OCQ are similar. Therefore,

$$\frac{OP}{OQ} = \frac{OA}{OC} = \frac{9/4}{3} = \frac{3}{4}$$

35. (2) The line passing through $(2, 3)$ and perpendicular to $-y + 3x + 4 = 0$ is

$$\frac{y-3}{x-2} = -\frac{1}{3}$$

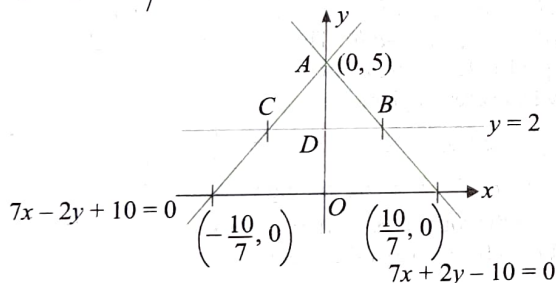
$$\text{or } 3y + x - 11 = 0$$

Therefore, the foot is $x = -1/10, y = 37/10$.

36. (3) We have

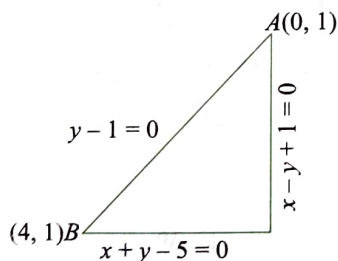
$$B \equiv \left(\frac{6}{7}, 2\right), C \equiv \left(-\frac{6}{7}, 2\right)$$

$$\text{or } BC = \frac{12}{7}, AD = 3$$



$$\therefore \Delta_{ABC} = \frac{1}{2} \times \frac{12}{7} \times 3 = \frac{18}{7} \text{ sq. units}$$

37. (1)



Since the triangle is right-angled, the circumcenter will be the middle point of hypotenuse, i.e., $(2, 1)$.

38. (1) The distance between $(2/m, 2)$ and $(6/m, 6)$ is less than 5. Hence,

$$\left(\frac{2}{m} - \frac{6}{m}\right)^2 + (2 - 6)^2 < 25$$

$$\text{or } \frac{16}{m^2} < 9$$

$$\text{or } m^2 > \frac{16}{9}$$

$$\text{i.e., } m > \frac{4}{3} \text{ or } m < -\frac{4}{3}$$

39. (2) We have line $x + y - 1 = 0$.

$$\text{Let } L(x, y) = x + y - 1.$$

$$\therefore L(3, 5) = 3 + 5 - 1 = 7 > 0$$

For points $(3, 5)$ and $(\sin \theta, \cos \theta)$ to lie on the same side of line $x + y - 1 = 0$, we must have

$$L(\cos \theta, \sin \theta) > 0$$

$$\text{or } \sin \theta + \cos \theta - 1 > 0$$

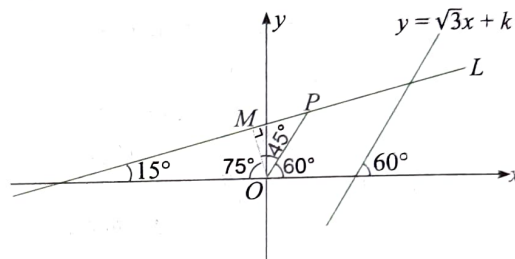
$$\sin\left(\frac{\pi}{4} + \theta\right) > \frac{1}{\sqrt{2}}$$

$$\frac{\pi}{4} < \frac{\pi}{4} + \theta < \frac{3\pi}{4}$$

$$0 < \theta < \frac{\pi}{2}$$

40. (4) Slope of the line $L \equiv (1 + \sqrt{3})y + (1 - \sqrt{3})x = 10$ is $\frac{\sqrt{3}-1}{\sqrt{3}+1} = \tan 15^\circ$.

Slope of line $y = \sqrt{3}x + k$ is $\sqrt{3} = \tan 60^\circ$.



In the figure, distance of origin from line L measured along the line $y = \sqrt{3}x + k$ is OP .

From the figure,

$$OP = OM \sec 45^\circ$$

$$= \frac{|0 + 0 - 10|}{\sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}} \sqrt{2} = \frac{10}{\sqrt{8}} \sqrt{2} = 5$$

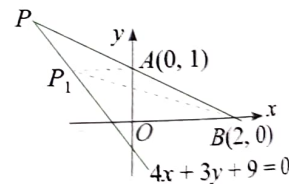
41. (2) The equation of AB is

$$y - 1 = \frac{0-1}{2-1}x \text{ or } x + 2y - 2 = 0$$

$$|PA - PB| \leq AB$$

Thus, $|PA - PB|$ is maximum if the points A , B , and P are collinear.

Hence, solving $x + 2y - 2 = 0$ and $4x + 3y + 9 = 0$, we get point $P \equiv (-24/5, 17/5)$.



42. (3) Consider a point A' , the image of A in $y = x$. Therefore, the coordinates of A' are $(4, 3)$.

Notice that A and B lie on the same side with respect to $y = x$. Then $PA = PA'$. Thus, $PA + PB$ is minimum, if $PA' + PB$ is minimum, i.e., if P , A' , B are collinear. Now, the equation of $A'B$ is

$$y - 3 = \frac{13-3}{7-4}(x - 4)$$

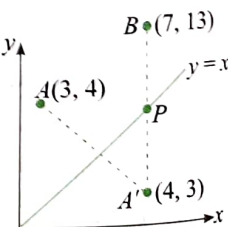
$$\text{or } 3y - 10x + 31 = 0$$

It intersects $y = x$ at $(31/7, 31/7)$, which is the required point P .

43. (3) The given inequality is equivalent to the following system of inequalities.

$$2x + 3y \leq 6, \text{ when } x \geq 0, y \geq 0$$

$$2x - 3y \leq 6, \text{ when } x \geq 0, y \leq 0$$



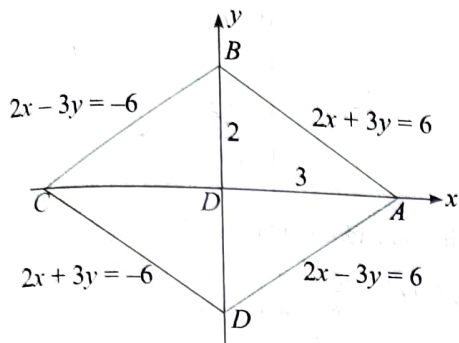
$$-2x + 3y \leq 6, \text{ when } x \leq 0, y \geq 0$$

$$-2x - 3y \leq 6, \text{ when } x \leq 0, y \leq 0$$

which represents a rhombus with sides

$$2x \pm 3y = 6 \text{ and } 2x \pm 3y = -6$$

The lengths of the diagonals are 6 units and 4 units, respectively, along the x - and y -axes. Therefore, the required area is $\frac{1}{2} \times 6 \times 4 = 12$ sq. units.



4. (4) As the altitude from A is fixed and the orthocenter lies on the altitude, $x + y = 3$ is the required locus.

5. (3) Incentre I is point of intersection of two angle bisectors which is $(2, 1)$.

$$IA = \sqrt{2^2 + 2^2} = 2\sqrt{2}$$

$$\text{Also, } AI = r \operatorname{cosec} \frac{A}{2}$$

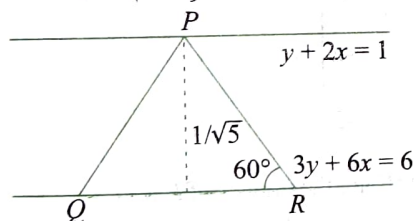
$$\angle BIC = \frac{\pi}{2} + \frac{A}{2}$$

$$\text{or } \tan\left(\frac{\pi}{2} + \frac{A}{2}\right) = \left|\frac{1-2}{1+2}\right|$$

$$\text{or } \tan \frac{A}{2} = 3$$

$$\text{So, } r = \frac{AI}{\operatorname{cosec} \frac{A}{2}} = \frac{2\sqrt{2}}{\sqrt{1+\frac{1}{9}}} \cdot \frac{2\sqrt{2} \times 3}{\sqrt{10}} = \frac{6}{\sqrt{5}}$$

46. (1)



The given lines are $y + 2x = 1$ and $y + 2x = 2$. The distance between the lines is $(2-1)/\sqrt{5} = 1/\sqrt{5}$.

The side length of the triangle is

$$\frac{1}{\sqrt{5}} \operatorname{cosec} 60^\circ = \frac{2}{\sqrt{15}}$$

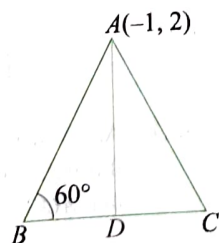
$$47. (1) AD = \left| \frac{-2-2-1}{\sqrt{(2)^2 + (-1)^2}} \right| = \left| \frac{-5}{\sqrt{5}} \right| = \sqrt{5}$$

$$\therefore \tan 60^\circ = \frac{AD}{BD}$$

$$\text{or } \sqrt{3} = \frac{\sqrt{5}}{BD}$$

$$\text{or } BD = \frac{\sqrt{5}}{\sqrt{3}}$$

$$\therefore BC = 2BD = 2\sqrt{\frac{5}{3}} = \sqrt{\frac{20}{3}}$$



48. (3) For any point $P(x, y)$ that is equidistant from the given line, we have

$$x + y - \sqrt{2} = -(x + y - 2\sqrt{2})$$

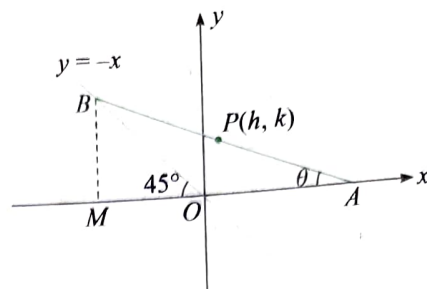
$$\text{or } 2x + 2y - 3\sqrt{2} = 0$$

49. (3) Since the diagonals are perpendicular, the given quadrilateral is a rhombus. So, the distances between two pairs of parallel sides are equal. Hence,

$$\left| \frac{c' - c}{\sqrt{a'^2 + b'^2}} \right| = \left| \frac{c' - c}{\sqrt{a^2 + b^2}} \right|$$

$$\text{or } a^2 + b^2 = a'^2 + b'^2$$

50. (1)



As shown in the figure, line segment AB , of fixed length 2 units, slides such that one of its ends A is on x -axis and other end B is on line $x + y = 0$.

Let $P(h, k)$ be midpoint of AB .

Also, let $\angle BAO = \theta$.

So, in triangle AMB , $BM = 2 \sin \theta$ and $AM = 2 \cos \theta$.

In triangle BMO , $MO = MB = 2 \sin \theta$

$$\therefore A = (2 \cos \theta - 2 \sin \theta, 0) \text{ and } B = (-2 \sin \theta, 2 \sin \theta)$$

Now, $P(h, k)$ is the midpoint of AB .

$$\therefore 2h = 2 \cos \theta - 2 \sin \theta + (-2 \sin \theta) \text{ and } 2k = 2 \sin \theta$$

$$\Rightarrow h = \cos \theta - 2 \sin \theta \text{ and } k = \sin \theta$$

$$\Rightarrow h + 2k = \cos \theta \text{ and } k = \sin \theta$$

Squaring and adding, we get

$$(h + 2k)^2 + k^2 = 1$$

$$\text{or } x^2 + 5y^2 + 4xy - 1 = 0$$

This is the required locus.

51. (2) Given vertices are $A(2a, 0)$ and $B(0, a)$

Let the coordinates of the third vertex C be $(2a, t)$. Now, $AC = BC$. Hence,

$$t = \sqrt{4a^2 + (a-t)^2} \text{ or } t = \frac{5a}{2}$$

So, the coordinates of the third vertex C are $(2a, 5a/2)$.

Therefore, area of the triangle is

$$\frac{1}{2} \begin{vmatrix} 2a & 5a/2 & 1 \\ 2a & 0 & 1 \\ 0 & a & 1 \end{vmatrix} = \frac{5a^2}{2} \text{ sq. units}$$

52. (4) Let $M \equiv (0, h)$. Then

$$\text{or } N \equiv (0, h + 4)$$

The equation of AM is

$$\frac{x}{-4} + \frac{y}{h} = 1$$

$$\text{or } \frac{y}{h} = \frac{4+x}{4}$$

$$\text{or } h = \frac{4y}{4+x}$$

The equation of BN is

$$\frac{x}{4} + \frac{y}{h+4} = 1$$

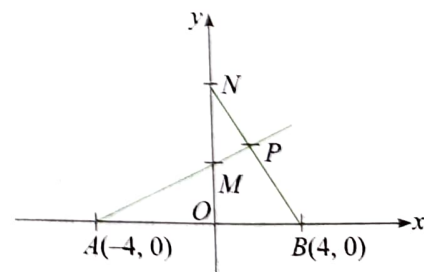
$$\text{or } \frac{y}{h+4} = \frac{4-x}{4}$$

$$\text{or } h + 4 = \frac{4y}{4-x}$$

$$\text{or } h = \frac{4y - 16 + 4x}{4-x}$$

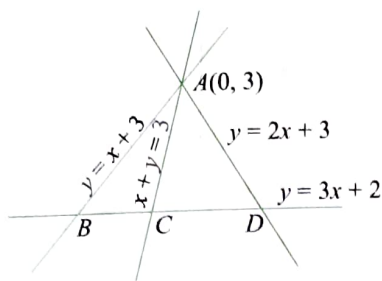
$$\text{or } \frac{4(y-4+x)}{4-x} = \frac{4y}{4+x} \text{ (Eliminating } h)$$

$$\text{or } 2xy - 16 + x^2 = 0, \text{ which is the required locus}$$



53. (3) The lines $y = x + 3$, $y = 2x + 3$ and $y + x = 3$ are concurrent at $(0, 3)$.

Rough sketch of the lines is as shown in the figure.



From the figure, three triangles ABD , ABC and ACD can be formed.

54. (3) Given line is $\frac{x}{a} + \frac{y}{b} = 1$

It meets the axes at points $A(a, 0)$ and $B(0, b)$.

Area of triangle AOB , $\Delta = \frac{1}{2}ab$

Given that 8 is H.M. of a and b .

$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{4}$$

$$\Rightarrow b = \frac{4a}{a-4}$$

$$\therefore \Delta = \frac{2a^2}{a-4}$$

Differentiating w.r.t. a , we get

$$\frac{d\Delta}{da} = 2 \frac{2a(a-4) - a^2}{(a-4)^2}$$

$$\Rightarrow \frac{d\Delta}{da} = \frac{2a(a-8)}{(a-4)^2}$$

If $\frac{d\Delta}{da} = 0$, then $a = 8$.

This is the value of a for which area is minimum.

$$\therefore \Delta_{\min} = 32 \text{ sq. unit.}$$

55. (4) Let the coordinates of C be $(1, c)$. Then,

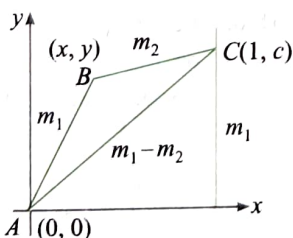
$$m_2 = \frac{c-y}{1-x}$$

$$\text{or } m_2 = \frac{c - m_1 x}{1 - x}$$

$$\text{or } m_2 - m_2 x = c - m_1 x$$

$$\text{or } (m_1 - m_2)x = c - m_2$$

$$\text{or } c = (m_1 - m_2)x + m_2 \quad (1)$$



Now, the area of ΔABC is

$$\frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ x & m_1 x & 1 \\ 1 & c & 1 \end{vmatrix} = \frac{1}{2} [cx - m_1 x]$$

$$= \frac{1}{2} [(m_1 - m_2)x + m_2]x - m_1 x$$

$$= \frac{1}{2} [(m_1 - m_2)x^2 + m_2 x - m_1 x]$$

$$= \frac{1}{2} (m_1 - m_2)(x - x^2)$$

$$[\because x > x^2 \text{ in } (0, 1)]$$

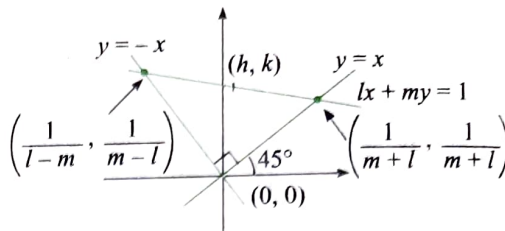
$$\text{Hence, } f(x) = \frac{1}{2}(x - x^2)$$

$$f(x)_{\max} = \frac{1}{8} \text{ when } x = \frac{1}{2}$$

56. (1) The coordinates of circumcenter are $(l/(l^2 - m^2), m/(m^2 - l^2))$. Hence,

$$h = \frac{l}{l^2 - m^2} \quad (1)$$

$$k = -\frac{m}{l^2 - m^2} \quad (2)$$



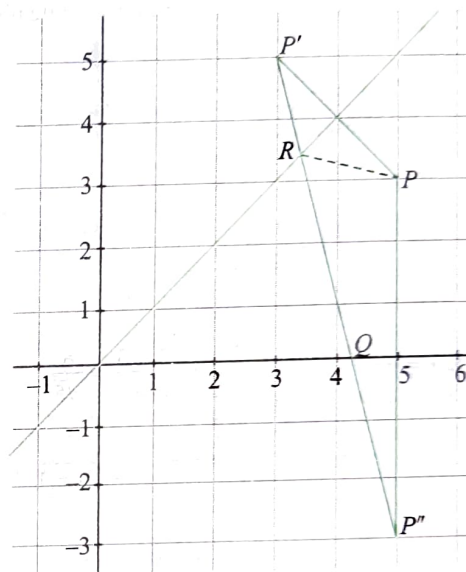
Squaring and adding (1) and (2), we get

$$h^2 + k^2 = \frac{l^2 + m^2}{(l^2 - m^2)^2} = \frac{1}{(l^2 - m^2)^2} \quad (\text{Putting } l^2 + m^2 = 1)$$

$$\text{Also, } h^2 - k^2 = \frac{1}{l^2 - m^2}$$

Therefore, the locus is $x^2 + y^2 = (x^2 - y^2)^2$.

57. (1)



P' and P'' are the images of P w.r.t. $y = x$ and $y = 0$, respectively.

Therefore, $P' \equiv (5, 3)$.

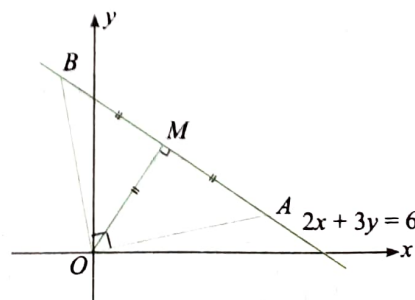
The equation of PP' is

$$y + 3 = \frac{5 + 3}{3 - 5}(x - 5)$$

$$\therefore x = 5 + 3\left(-\frac{1}{4}\right) = \frac{17}{4}$$

$$\therefore Q \equiv \left(\frac{17}{4}, 0\right)$$

58. (1) As shown in the following figure, pair of perpendicular lines through origin meets given line $2x + 3y = 6$ at A and B such that AOB is right angled isosceles triangle.



Now, $OM = BM = AM$, where $OM = \frac{|0+0-6|}{\sqrt{2^2+3^2}} = \frac{6}{\sqrt{13}}$

$\therefore AB = 2AM = 2OM = \frac{12}{\sqrt{13}}$

$\therefore$ Area of triangle $AOB = \frac{1}{2} \times \frac{12}{\sqrt{13}} \times \frac{6}{\sqrt{13}} = \frac{36}{13}$ sq. units.

(2) Sum of distances of P from lines $2x - y - 3 = 0$ and $x + 3y + 4 = 0$ is 7.

$\therefore \frac{|2x - y - 3|}{\sqrt{5}} + \frac{|x + 3y + 4|}{\sqrt{10}} = 7$

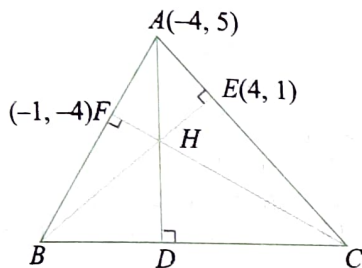
or $\pm \sqrt{2}(2x - y - 3) \pm (x + 3y + 4) = 7\sqrt{10}$

The area of parallelogram formed by the lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + d_1 = 0$, $a_1x + b_1y + c_2 = 0$ and $a_2x + b_2y + d_2 = 0$ is

$\left| \frac{(c_1 - c_2)(d_1 - d_2)}{a_1b_2 - a_2b_1} \right|$ sq. units.

So, area = $\frac{14^2 \times 10}{14\sqrt{2}} = 70\sqrt{2}$

(3)



Slope of $AB = \frac{5 - (-4)}{-4 - 1} = -3$

$\therefore$ Equation of $AB \equiv 3x + y + 7 = 0$

Slope of $CF = \frac{1}{3}$

$\therefore$ Equation of $CF \equiv x - 3y - 11 = 0$

Slope of $AC = \frac{5 - 1}{-4 - 4} = -\frac{1}{2}$

$\therefore$ Equation of $AC \equiv x + 2y - 6 = 0$

Slope of $BE = 2$

$\therefore$ Equation of $BE \equiv 2x - y - 7 = 0$

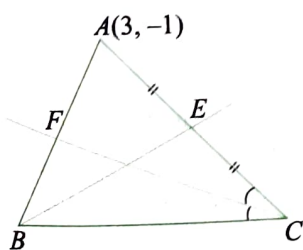
Solving (1) and (4), we get $B \equiv (0, -7)$.

Solving (2) and (3), we get $C \equiv (8, -1)$.

Slope of $BC = \frac{-7 - 1}{0 - 8} = \frac{3}{4}$

$\therefore$ Equation of $BC \equiv 3x - 4y - 28 = 0$

(2)



Point E is on the median $x - 4y + 10 = 0$.

$\therefore E \equiv \left(x_1, \frac{x_1 + 10}{4} \right)$

Point C is on the bisector $6x + 10y - 59 = 0$.

$\therefore C \equiv \left(x_2, \frac{59 - 6x_2}{10} \right)$

Midpoint of AC is E .

$\therefore \frac{x_2 + 3}{2} = x_1$

$\Rightarrow x_2 = 2x_1 - 3$

and $\frac{59 - 6x_2 - 1}{10} = \frac{x_1 + 10}{4}$

$\Rightarrow \frac{49 - 6(2x_1 - 3)}{20} = \frac{x_1 + 10}{4}$

$\Rightarrow 49 - 12x_1 + 18 = 5x_1 + 50$

$\Rightarrow 17x_1 = 17$

$\therefore x_1 = 1$

Coordinates of E are $\left(1, \frac{11}{4} \right)$.

Equation of AC is

$(y + 1) = \frac{\frac{11}{4} + 1}{-2}(x - 3)$

$\Rightarrow -8y - 8 = 15x - 45$

$\Rightarrow 15x + 8y = 37$

62. (1) From the given information, it is clear that midpoint of AB is the foot of the perpendicular from $P(1, 2)$ on the line $2x - y + 3 = 0$, which is given by

$\frac{x - 1}{2} = \frac{y - 2}{-1} = -\frac{2(1) - 2 + 3}{(2)^2 + (-1)^2}$

or $(x, y) \equiv \left(\frac{-1}{5}, \frac{13}{5} \right)$

63. (4) The given lines are

$(a + b)x + (a - b)y - 2ab = 0$ (1)

$(a - b)x + (a + b)y - 2ab = 0$ (2)

and $x + y = 0$ (3)

Lines (1) and (2) are symmetrical about the line $y = x$ (as on interchanging x and y in 1st line gives 2nd line).

So, $y = x$ is one of the angle bisectors.

This is perpendicular to the third side $x + y = 0$.

Therefore, triangle is isosceles.

64. (3) $ax + by = 1$ will be one of the bisectors of the given line.

Bisectors of the given lines are

$\frac{3x + 4y - 5}{5} = \pm \left(\frac{5x - 12y - 10}{13} \right)$

$\Rightarrow 64x - 8y = 115$ or $14x + 112y = 15$

$\therefore a = \frac{64}{115}, b = -\frac{8}{115}$ or $a = \frac{14}{15}, b = \frac{112}{15}$

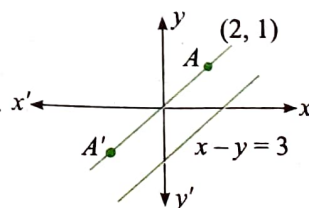
65. (3) Since the point $A(2, 1)$ is translated parallel to $x - y = 3$, AA' has the same slope as that of $x - y = 3$. Therefore, AA' passes through $(2, 1)$ and has slope 1. Here, $\tan \theta = 1$ or $\cos \theta = 1/\sqrt{2}$, $\sin \theta = 1/\sqrt{2}$.

Thus, the equation of AA' is

$\frac{x - 2}{\cos(\pi/4)} = \frac{y - 1}{\sin(\pi/4)}$

Since $AA' = 4$, the coordinates of A' are given by

$\frac{x - 2}{\cos(\pi/4)} = \frac{y - 1}{\sin(\pi/4)} = -4$



$$\text{or } x = 2 - 4 \cos \frac{\pi}{4}, y = 1 - 4 \sin \frac{\pi}{4}$$

$$\text{or } x = 2 - 2\sqrt{2}, y = 1 - 2\sqrt{2}$$

Hence, the coordinates of A' are $(2 - 2\sqrt{2}, 1 - 2\sqrt{2})$.

66. (1) The extremities of the given diagonal are $(4, 0)$ and $(0, 6)$. Hence, the slope of this diagonal is $-3/2$ and the slope of other diagonal is $2/3$. The equation of the other diagonal is

$$\frac{x-2}{3\sqrt{13}} = \frac{y-3}{2\sqrt{13}} = r$$

For the extremities of the diagonal, $r = \pm\sqrt{13}$. Hence,

$$x - 2 = \pm 3, y - 3 = \pm 2$$

$$x = 5, -1 \text{ and } y = 5, 1$$

Therefore, the extremities of the diagonal are $(5, 5)$ and $(-1, 1)$.

67. (1) Slope of given line $x + y - 1 = 0$ is

$$\tan \theta = -1$$

$\therefore \theta = 135^\circ$, which is inclination of line with x -axis.

Point $P(2, 1)$ is on the line.

Coordinates of point Q on the line at distance $3\sqrt{2}$ from it with decreasing ordinate are

$$(2 - 3\sqrt{2} \cos 135^\circ, 1 - 3\sqrt{2} \sin 135^\circ)$$

$$= \left(2 - 3\sqrt{2} \left(-\frac{1}{\sqrt{2}} \right), 1 - 3\sqrt{2} \left(\frac{1}{\sqrt{2}} \right) \right)$$

$$= (5, -2)$$

If (h, k) is image of point Q in the line $x + y - 1 = 0$, then

$$\frac{h-5}{1} = \frac{k+2}{1} = \frac{-2(5-2-1)}{2}$$

$$\therefore h = 3, k = -4$$

68. (1) Since $xy > 0$, P lies either in the first quadrant or in the third quadrant. The inequality $x + y < 1$ represents all the points below the line $x + y = 1$ so that $xy > 0$ and $x + y < 1$ imply that P lies either inside the triangle OAB or in the third quadrant.

69. (2) The reflections of A in the two angle bisectors will lie on the line BC . So, $(2, 1)$ and $(1, -2)$ will lie on BC . The equation of BC will be

$$y + 2 = \left(\frac{1+2}{2-1} \right)(x-1)$$

$$\text{i.e., } 3x - y = 5$$

70. (2) Lines $x \cos \alpha + y \sin \alpha = p$ and $x \sin \alpha - y \cos \alpha = 0$ are mutually perpendicular. Thus, $ax + by + p = 0$ will be equally inclined to these lines and would be the angle bisector of these lines. Now, the equations of angle bisectors are

$$x \sin \alpha - y \cos \alpha = \pm (x \cos \alpha + y \sin \alpha - p)$$

$$\text{or } x(\cos \alpha - \sin \alpha) + y(\sin \alpha + \cos \alpha) = p$$

$$\text{or } x(\sin \alpha + \cos \alpha) - y(\cos \alpha - \sin \alpha) = p$$

Comparing these lines with $ax + by + p = 0$, we get

$$\frac{a}{\cos \alpha - \sin \alpha} = \frac{b}{\sin \alpha + \cos \alpha} = 1$$

$$\Rightarrow a^2 + b^2 = 2$$

$$\Rightarrow \frac{a}{\sin \alpha + \cos \alpha} = \frac{b}{\sin \alpha - \cos \alpha} = 1$$

$$\Rightarrow a^2 + b^2 = 2$$

71. (3) We have to find the locus of the point (h, k) whose image on the line $2x - y - 1 = 0$ lies on the line $y = x$. Now, the image of (h, k) on the line $2x - y - 1 = 0$ is given by

$$\frac{x_2 - h}{2} = \frac{y_2 - k}{-1} = -\frac{2(2h - k - 1)}{5}$$

$$\text{or } x_2 = \frac{-3h + 4k + 4}{5}$$

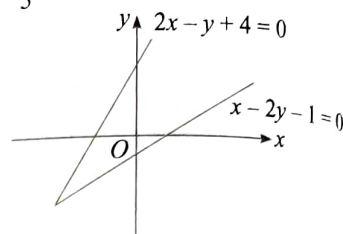
$$\text{and } y_2 = \frac{4h + 3k - 2}{5}$$

This point lies on $y = x$. Then,

$$\frac{-3h + 4k + 4}{5} = \frac{4h + 3k - 2}{5} \text{ or } 7h - k = 6$$

72. (2) Clearly, from the figure, the origin is contained in the acute angle. Writing the equations of the lines as $2x - y + 4 = 0$ and $-x + 2y + 1 = 0$, the required bisector is

$$\frac{2x - y + 4}{\sqrt{5}} = \frac{-x + 2y + 1}{\sqrt{5}}$$



73. (2) The set of lines is $4ax + 3by + c = 0$, where $a + b + c = 0$. Eliminating c , we get

$$4ax + 3by - (a + b) = 0$$

$$\text{or } a(4x - 1) + b(3y - 1) = 0$$

This passes through the intersection of the lines $4x - 1 = 0$ and $3y - 1 = 0$, i.e., $x = 1/4, y = 1/3$, i.e., $(1/4, 1/3)$.

74. (2) If the given lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\text{or } \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

$$(\text{Applying } C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1)$$

$$\text{or } a(b-1)(c-1) - (c-1)(1-a) - (b-1)(1-a) = 0$$

$$\text{or } \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

$$[\text{Dividing by } (1-a)(1-b)(1-c)]$$

Adding 1 on both sides, we get

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

75. (3) The lines are concurrent if

$$\begin{vmatrix} 1 & 2 & -1 \\ a & 1 & 3 \\ b & -1 & 2 \end{vmatrix} = 0$$

$$\text{or } 7b - 3a + 5 = 0$$

The locus of (a, b) is $3x - 7y = 5$.

Least distance from $(0, 0) = \text{Length of perpendicular from } (0, 0) = \frac{5}{\sqrt{58}}$

76. (1) The three lines are concurrent if

$$\begin{vmatrix} 1 & 2 & -9 \\ 3 & 5 & -5 \\ a & b & -1 \end{vmatrix} = 0$$

$$\text{or } 35a - 22b + 1 = 0$$

which is true if the line $35x - 22y + 1 = 0$ passes through (a, b) .

77. (3) The equation of AO is $2x + 3y - 1 + \lambda(x + 2y - 1) = 0$, where $\lambda = -1$, since the line passes through the origin. So, $x + y = 0$. Since AO is perpendicular to BC , we have

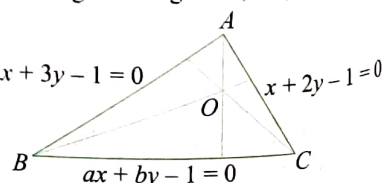
$$(-1)\left(-\frac{a}{b}\right) = -1$$

$$\therefore a = -b$$

Similarly,

$$(2x + 3y - 1) + \mu(ax - ay - 1) = 0$$

will be the equation of BO for $\mu = -1$. Now, BO is perpendicular to AC . Hence,



$$\left\{ -\frac{(2-a)}{3+a} \right\} \left(-\frac{1}{2} \right) = -1$$

$$\therefore a = -8, b = 8$$

$$\frac{a}{\sqrt{bc}} - 2 = \sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}$$

$$\text{or } a = b + c + 2\sqrt{bc}$$

$$\text{or } a = (\sqrt{b} + \sqrt{c})^2$$

$$\text{or } (\sqrt{a} - \sqrt{b} - \sqrt{c})(\sqrt{a} + \sqrt{b} + \sqrt{c}) = 0$$

$$\text{or } \sqrt{a} - \sqrt{b} - \sqrt{c} = 0$$

$$\text{since } \sqrt{a} + \sqrt{b} + \sqrt{c} \neq 0 \text{ (as } a, b, c > 0 \text{),}$$

$$\text{Comparing with } \sqrt{ax} + \sqrt{by} + \sqrt{c} = 0, \text{ we have } x = -1, y = 1.$$

80. (1) The first two families of lines pass through (1, 1) and (3, 3), respectively. The point of intersection of the lines belonging to the third family of lines will lie on line $y = x$. Hence,

$$ax + x - 2 = 0 \text{ and } 6x + ax - a = 0$$

$$\text{or } \frac{2}{a+1} = \frac{a}{6+a}$$

$$\text{or } a^2 - a - 12 = 0 \text{ or } (a-4)(a+3) = 0$$

80. (1) Given equation of family of lines is

$$(2x + y - 3) + \lambda(x + 2y - 3) = 0, \lambda \in \mathbb{R}.$$

The family of lines is concurrent at point of intersection of $2x + y - 3 = 0$ and $x + 2y - 3 = 0$ which is $A(1, 1)$.

Two members L_1 and L_2 of this family and line $x + y = 0$ form an equilateral triangle.

Let foot of perpendicular from $A(1, 1)$ on the line $x + y = 0$ be $M(h, k)$.

$$\frac{h-1}{1} = \frac{k-1}{1} = \frac{-(1+1)}{2}$$

$$\therefore M \equiv (h, k) \equiv (0, 0)$$

Since triangle is equilateral, AM is median, altitude and angle bisector.

$$AM = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$$

Incentre coincides with centroid G such that $AG = \frac{2}{3}\sqrt{2}$.

Slope of AM is $1 = \tan 45^\circ$.

$$\therefore G \equiv \left(1 - \frac{2\sqrt{2}}{3} \cos 45^\circ, 1 - \frac{2\sqrt{2}}{3} \sin 45^\circ \right)$$

$$\equiv \left(\frac{1}{3}, \frac{1}{3} \right)$$

81. (2) Since $a \in (0, 1)$

$$\Rightarrow \sin^{-1} \frac{1}{\sqrt{1+a^2}} = \cot^{-1} a = \frac{\pi}{2} - \tan^{-1} a$$

$$\text{Now, } x \tan^{-1} a + y \sin^{-1} \frac{1}{\sqrt{1+a^2}} + 2 = 0$$

$$\Rightarrow x \tan^{-1} a + y \left(\frac{\pi}{2} - \tan^{-1} a \right) + 2 = 0$$

$$\Rightarrow (x-y) \tan^{-1} a + \left(\frac{\pi y}{2} + 2 \right) = 0$$

$$\text{Clearly, this passes through } \left(-\frac{4}{\pi}, -\frac{4}{\pi} \right).$$

$$82. (3) \sin(\alpha + \beta) \sin(\alpha - \beta) = \sin \gamma (2 \sin \beta + \sin \gamma)$$

$$\text{or } \sin^2 \alpha - (\sin \beta + \sin \gamma)^2 = 0$$

$$\text{or } (\sin \alpha + \sin \beta + \sin \gamma)(\sin \alpha - \sin \beta - \sin \gamma) = 0$$

Since $0 < \alpha, \beta, \gamma < \pi$, we have

$$\sin \alpha + \sin \beta + \sin \gamma \neq 0$$

$$\therefore \sin \alpha - \sin \beta - \sin \gamma = 0$$

So, $x \sin \alpha + y \sin \beta - \sin \gamma = 0$ passes through the fixed point (1, -1).

Multiple Correct Answers Type

1. (1), (3)

Let $L = 3x - 4y - 8$. Then the value of L at (3, 4) is $3 \times 3 - 4 \times 4 - 8 = -15 < 0$. Hence, for the point $P(x, y)$, we should have

$$L > 0$$

$$\text{or } 3x - 4y - 8 > 0$$

$$\text{or } 3x - 4(-3x) - 8 > 0$$

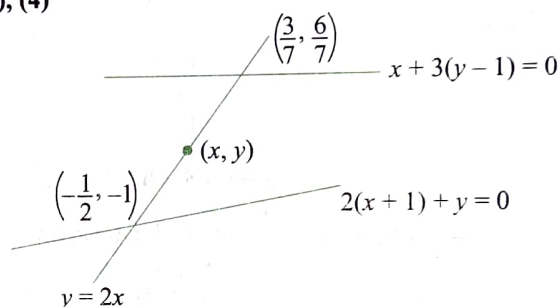
$$[\because P(x, y) \text{ lies on } y = -3x]$$

$$\text{or } x > \frac{8}{15}$$

$$\text{and } -y - 4y - 8 > 0$$

$$\text{or } y < -\frac{8}{5}$$

2. (2), (4)



Solving $y = 2x$, $2(x+1) + y = 0$, we get $x = -1/2$, $y = -1$. Solving $y = 2x$, $x + 3(y-1) = 0$, we get $x = 3/7$, $y = 6/7$.

3. (1), (2)

The equations of lines along OA , OB , and AB are $y = 0$, $x = 0$, and $x + y = \sqrt{3}/2$, respectively.

Now, P and B will lie on the same side of $y = 0$ if $\cos \theta > 0$. Similarly, P and A will lie on the same side of $x = 0$ if $\sin \theta > 0$, and P and O will

lie on the same side of $x + y = \sqrt{3}/2$ if $\sin \theta + \cos \theta < \sqrt{3}/2$. Hence, P will lie inside $\triangle ABC$ if $\sin \theta > 0$, $\cos \theta > 0$, and $\sin \theta + \cos \theta < \sqrt{3}/2$. Now,

$$\sin \theta + \cos \theta < \sqrt{\frac{3}{2}}$$

$$\text{or } \sin\left(\theta + \frac{\pi}{4}\right) < \sqrt{\frac{3}{4}}$$

Since $\sin \theta > 0$ and $\cos \theta > 0$, $0 < \theta < \pi/2$ or $5\pi/12 < \theta < \pi/2$.

4. (1), (4)

Distance between $x + 2y + 3 = 0$ and $x + 2y - 7 = 0$ is $10/\sqrt{5}$. Let the remaining side parallel to $2x - y - 4 = 0$ be $2x - y + \lambda = 0$. We have

$$\frac{|\lambda + 4|}{\sqrt{5}} = \frac{10}{\sqrt{5}} \text{ or } \lambda = 6, -14$$

Thus, the remaining side is $2x - y + 6 = 0$ or $2x - y - 14 = 0$.

5. (2), (4)

Let the angle be θ . Then, the equation of the given line is

$$\frac{x-1}{\cos \theta} = \frac{y-2}{\sin \theta}$$

(i)

The coordinates of a point on (i) at a distance $\sqrt{6}/3$ from $(1, 2)$ are $(1 + \sqrt{6}/3 \cos \theta, 2 + \sqrt{6}/3 \sin \theta)$. This point lies on $x + y = 4$. Therefore,

$$1 + \frac{\sqrt{6}}{3} \cos \theta + 2 + \frac{\sqrt{6}}{3} \sin \theta = 4$$

$$\text{or } \cos \theta + \sin \theta = \sqrt{\frac{3}{2}}$$

$$\text{or } \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta = \frac{\sqrt{3}}{2}$$

$$\text{or } \cos\left(\theta - \frac{\pi}{4}\right) = \cos\left(\pm \frac{\pi}{6}\right)$$

$$\text{or } \theta - \frac{\pi}{4} = \pm \frac{\pi}{6}$$

$$\text{i.e., } \theta = 75^\circ \text{ or } \theta = 15^\circ$$

6. (2), (3)

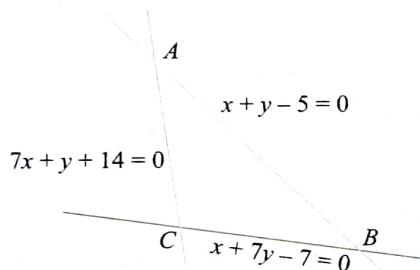
For the two lines $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$, the angle bisectors are given by

$$\frac{24x + 7y - 20}{25} = \pm \frac{4x - 3y - 2}{5}$$

Taking positive sign, we get

$$2x + 11y - 5 = 0$$

7. (1), (3), (4)



The slopes of the lines AB , BC and CA are -1 , $-\frac{1}{7}$ and -7 , respectively

$$\text{Let } m_1 = -\frac{1}{7}, m_2 = -1, m_3 = -7.$$

$$\therefore m_1 > m_2 > m_3$$

So, tangent of internal angles of the triangle are

$$\tan A = \frac{3}{4}, \tan B = \frac{3}{4} \text{ and } \tan C = -\frac{24}{7}$$

So, interior angles A and B are acute and interior angle C is obtuse.

$\therefore$ Internal bisector of $B \equiv$ Acute angle bisector at B

$\equiv$ Acute angle bisector of lines AB and BC

$$\equiv 3x + 6y - 16 = 0$$

External bisector of $C \equiv$ Acute bisector of C

$\equiv$ Acute angle bisector of lines AC and BC

$$\equiv 8x + 8y + 7 = 0$$

8. (1), (2), (4)

Since the given points lie on the line $lx + my + n = 0$, a, b, c are the roots of the equation

$$l\left(\frac{t^3}{t-1}\right) + m\left(\frac{t^2-3}{t-1}\right) + n = 0$$

$$\text{or } lt^3 + mt^2 + nt - (3m + n) = 0 \quad (1)$$

$$\text{Hence, } a + b + c = -\frac{m}{l}$$

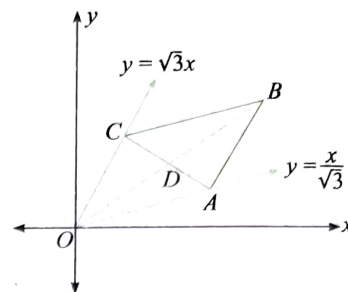
$$ab + bc + ca = \frac{n}{l} \quad (2)$$

$$abc = \frac{3m + n}{l} \quad (3)$$

So, from (1), (2), and (3), we get

$$abc - (bc + ca + ab) + 3(a + b + c) = 0$$

9. (1), (2)



Here, $\angle COA = 30^\circ$.

$$\text{Area of rhombus} = 2 \times \frac{1}{2} \times OA \times OC \sin 30^\circ$$

$$\text{or } 2 = \frac{1}{2} x^2$$

$$\text{or } OA = OC = 2$$

Also, $\angle OAB = 150^\circ$

$$\therefore \cos 150^\circ = \frac{OA^2 + AB^2 - OB^2}{2 OA \times AB}$$

$$OB^2 = 8 + 4\sqrt{3} \text{ or } OB = \sqrt{2}(\sqrt{3} + 1)$$

Hence, the coordinates of B are $(\pm\sqrt{2}(\sqrt{3} + 1) \cos 45^\circ, \pm\sqrt{2}(\sqrt{3} + 1) \sin 45^\circ)$.

10. (1), (2), (3), (4)

If points A, B, C, D are concyclic, then $ac = bd$. The coordinates of the points of intersection of lines are

$$\left(\frac{ac(b-d)}{bc-ad}, \frac{bd(c-a)}{bc-ad}\right)$$

Let the coordinates of the point of intersection be (h, k) . Then

$$h = \frac{ac(b-d)}{bc-ad}, k = \frac{bd(c-a)}{bc-ad}$$

Given $c^2 + a^2 = b^2 + d^2$. Since $ac = bd$, we have

$$(c-a)^2 = (b-d)^2$$

$$\text{or } (c-a) = \pm(b-d)$$

Then the locus of the points of intersection is $y = \pm x$.

11. (1), (4)

$$AB = 5, D \equiv \left(2, \frac{3}{2}\right)$$

$$CD = 5 \times \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$$

$$\text{Slope of } AB = -\frac{3}{4}$$

$$\text{Slope of } CD = \frac{4}{3}$$

If $C \equiv (h, k)$, then

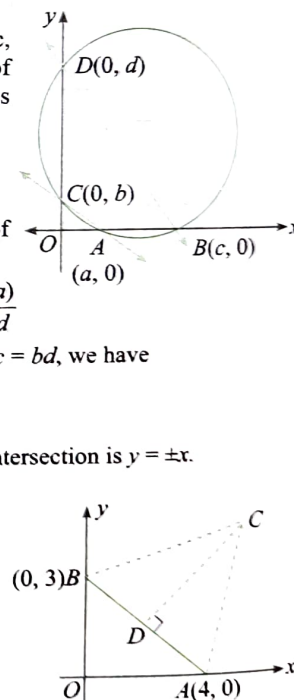
$$\frac{h-2}{3/5} = \frac{k-3/2}{4/5} = \pm \frac{5\sqrt{3}}{2}$$

$$\text{or } h = 2\left(1 - \frac{3\sqrt{3}}{4}\right), k = \frac{3}{2}\left(1 - \frac{4}{\sqrt{3}}\right)$$

$$\text{or } h = 2\left(1 + \frac{3\sqrt{3}}{4}\right), k = \frac{3}{2}\left(1 + \frac{4}{\sqrt{3}}\right)$$

12. (1), (3)

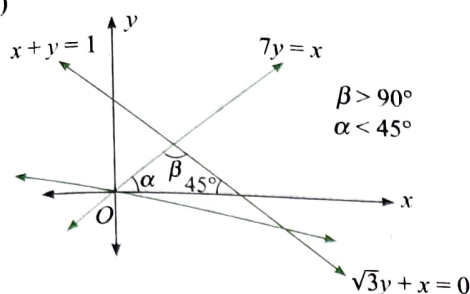
The equations of lines passing through $(1, 0)$ are given by $y = m(x-1)$. Its distance from the origin is $\sqrt{3}/2$. Hence,



$$\left| \frac{-m}{\sqrt{1+m^2}} \right| = \frac{\sqrt{3}}{2} \text{ or } m = \pm\sqrt{3}$$

Hence, the lines are $\sqrt{3}x + y - \sqrt{3} = 0$ and $\sqrt{3}x - y - \sqrt{3} = 0$.

13. (2), (3)



Clearly, from the figure, the triangle is obtuse-angled. Hence, the centroid and incenter lie inside the triangle. The orthocenter and circumcenter lie outside the triangle.

Therefore, it is an obtuse-angled triangle.

14. (1), (2)

The area of the triangle is given by

$$\frac{1}{2} \times \frac{2b}{a} \times \frac{2b}{c} = \frac{2b^2}{ac} = 2$$

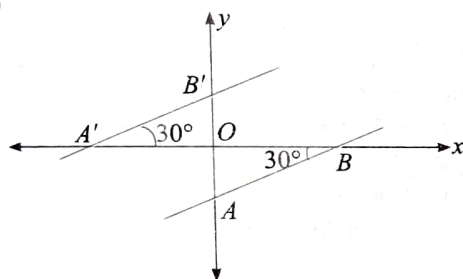
$$\text{or } b^2 = ac$$

Therefore, a, b, c are in GP. So, $a, -b, c$ are also in GP.

15. (2), (3)

Draw the diagram and verify.

16. (2), (4)



According to the question, $AB = 10$. So, $OA = 10 \sin 30^\circ = 5$. Then the equation of line is

$$y = \frac{1}{\sqrt{3}}x + 5$$

$$\text{or } x - \sqrt{3}y \pm 5\sqrt{3} = 0$$

17. (3), (4)

If the lines $x + y - 1 = 0$, $(m-1)x + (m^2-7)y - 5 = 0$, and $(m-2)x + (2m-5)y = 0$ are concurrent, then

$$\Delta = 0$$

$$\text{or } \begin{vmatrix} 1 & 1 & -1 \\ m-1 & m^2-7 & -5 \\ m-2 & 2m-5 & 0 \end{vmatrix} = 0$$

$$\text{or } (m-2)(-5+m^2-7) - (2m-5)(-5+m-1) + 0 = 0$$

$$\text{or } (m-2)(m^2-12) - (2m-5)(m-6) = 0$$

$$\text{or } m^3 - 4m^2 + 5m - 6 = 0$$

$$\text{or } (m-3)(m^2-m+2) = 0$$

$$\text{or } m = 3$$

If $m = 3$, then the two lines are parallel.

18. (2), (3)

Let the slope of required line be m . Then

$$\frac{1}{2} = \left| \frac{m - (-2)}{1 + (-2)m} \right|$$

$$\text{or } m = -\frac{3}{4} \text{ and } \infty$$

Hence, the equation of line is $y - 3 = -(3/4)(x - 2)$ and $x = 2$.

19. (1), (2)

Let p be the length of the perpendicular from the origin on the given line. Then its equation in normal form is

$$x \cos 30^\circ + y \sin 30^\circ = p$$

$$\text{or } \sqrt{3}x + y = 2p$$

This meets the coordinate axes at $A(2p/\sqrt{3}, 0)$ and $B(0, 2p)$. Therefore, the area of $\triangle AOB$ is

$$\frac{1}{2} \left(\frac{2p}{\sqrt{3}} \right) 2p = \frac{2p^2}{\sqrt{3}}$$

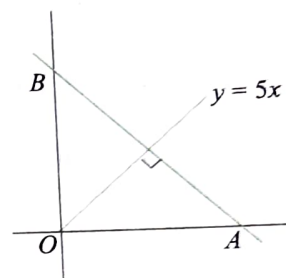
By hypothesis,

$$\frac{2p^2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ or } p = \pm 5$$

Hence, the lines are $\sqrt{3}x + y \pm 10 = 0$.

20. (1), (2)

Let the equation of line be $\frac{x}{a} + \frac{y}{b} = 1$.



AB is perpendicular to $y = 5x$.

$$\therefore -\frac{b}{a} \times 5 = -1 \text{ or } 5b = a$$

$$\text{Area of } \triangle OAB = \frac{1}{2} |ab|$$

$$\Rightarrow 10 = \frac{1}{2} |5b^2|$$

$$\Rightarrow b^2 = 4$$

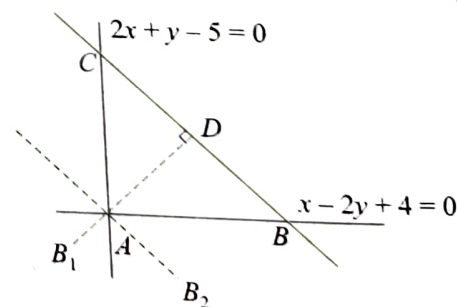
$$\Rightarrow b = \pm 2$$

When $b = 2$, $a = 10$ and when $b = -2$, $a = -10$.

The line can be $\frac{x}{10} + \frac{y}{2} = 1$ or $\frac{x}{10} + \frac{y}{2} = -1$.

21. (1), (2), (3), (4)

Given lines are mutually perpendicular and intersect at $A\left(\frac{6}{5}, \frac{13}{5}\right)$



Equation of angle bisectors B_1 and B_2 of the given lines are

$$x - 2y + 4 = \pm(2x + y - 5)$$

$$\text{or } x + 3y = 9 \text{ and } 3x - y = 1$$

Side BC will be parallel to these bisectors.

Let $AD = a$.

$$\therefore AB = a\sqrt{2}$$

$$\text{Area of triangle } ABC = \frac{1}{2} \times (a\sqrt{2})^2 = a^2$$

$$\therefore a^2 = 10$$

$$\text{or, } a = \sqrt{10}$$

Let equation of BC be $x + 3y = \lambda$. Then

$$\sqrt{10} = \frac{\left| \frac{6}{5} + \frac{39}{5} - \lambda \right|}{\sqrt{10}}$$

$$\lambda = -1, 19$$

So, equation of BC is $x + 3y = -1$ or $x + 3y = 19$.

If equation of BC is $3x - y = \lambda$, then

$$\sqrt{10} = \frac{\left| \frac{18}{5} - \frac{13}{5} - \lambda \right|}{\sqrt{10}}$$

$$\lambda = -9, 11$$

Therefore, equation of BC is $3x - y = -9$ or $3x - y = 11$

22. (1), (2), (3), (4)

First and third line intersect at $(0, 1)$.

This point also satisfies the second line for all real values of a .

23. (1), (2), (3)

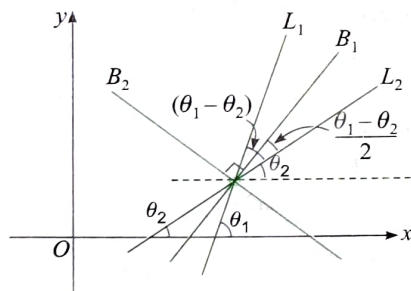
For the concurrency of three lines $px + qy + r = 0$, $qx + ry + p = 0$, $rx + py + q = 0$, we must have

$$\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$$

$$\text{or } 3pqr - p^3 - q^3 - r^3 = 0$$

$$\text{or } (p + q + r)(p^2 + q^2 + r^2 - pq - pr - rq) = 0$$

24. (1), (4)



From the figure, angle bisectors B_1 and B_2 have inclinations of

$$\frac{\theta_1 + \theta_2}{2} \text{ and } \left(\frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2} \right) \text{ with the } x\text{-axis.}$$

Therefore, equations of angle bisectors are

$$\frac{x - x_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}$$

$$\text{and } \frac{x - x_1}{\cos\left(\frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\sin\left(\frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2}\right)}$$

$$\text{or } \frac{x - x_1}{-\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$$

25. (1), (2)

We have $L_1 \equiv 3x - 4y + 2 = 0$

$$L_2 \equiv 4x - 3y + 5 = 0$$

$$\text{Here, } a_1a_2 + b_1b_2 = (3)(4) + (-4)(-3) = 24 > 0$$

So, acute angle bisector is

$$3x - 4y + 2 = 4x - 3y + 5$$

$$\text{or } x + y + 3 = 0$$

This bisector goes through the regions where expressions $3x - 4y + 2$ and $4x - 3y + 5$ have the same sign.

Also, this is the bisector of angle which contains origin.

26. (1), (3)

It is clear that the diagonals of the rhombus will be parallel to the bisectors of the given lines and will pass through $(1, 3)$. The equations of bisectors of the given lines are

$$\frac{x + y - 1}{\sqrt{2}} = \pm \left(\frac{7x - y - 5}{5\sqrt{2}} \right)$$

$$\text{or } 2x - 6y = 0, 6x + 2y = 5$$

Therefore, the equations of diagonals are $x - 3y + 8 = 0$ and $3x + y - 6 = 0$. Thus, the required vertex will be the point where these lines meet the line $y = 2x$. Solving these lines, we get the possible coordinates as $(8/5, 16/5)$ and $(6/5, 12/5)$.

27. (1), (2), (3), (4)

Let the slope of $u = 0$ be m . Then slope of $v = 0$ is $9m/2$. Therefore,

$$\frac{7}{9} = \left| \frac{m - \frac{9m}{2}}{1 + m \times \frac{9m}{2}} \right| = \left| \frac{-7m}{2 + 9m^2} \right|$$

$$\text{or } 9m^2 - 9m + 2 = 0 \text{ or } 9m^2 + 9m + 2 = 0$$

$$m = \frac{9 \pm \sqrt{81 - 72}}{18} = \frac{9 \pm 3}{18} = \frac{2}{3}, \frac{1}{3}$$

$$\text{or } m = \frac{-9 \pm 3}{18} = -\frac{2}{3}, -\frac{1}{3}$$

Therefore, the equations of the lines are

$$(i) \ 3y = x \text{ and } 2y = 3x$$

$$(ii) \ 3y = 2x \text{ and } y = 3x$$

$$(iii) \ x + 3y = 0 \text{ and } 3x + 2y = 0$$

$$(iv) \ 2x + 3y = 0 \text{ and } 3x + y = 0$$

28. (1), (4)

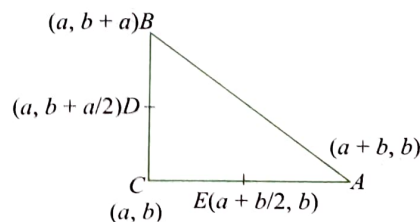
Note that the lines are perpendicular. Assume the coordinate axes to be directed along $u = 0$ and $v = 0$. Now, the lines $k_1u - k_2v = 0$ and $k_1u + k_2v = 0$ are equally inclined with the u - v axes. Hence, the bisectors are $u = 0$ and $v = 0$.

29. (1), (4)

$$m_1 = \text{Slope of } AD = -\frac{a}{2b}$$

$$m_2 = \text{Slope of } BE = -\frac{2a}{b}$$

$$\therefore \frac{m_1}{m_2} = \frac{1}{4}$$



$$\text{If } m_1 = 2, \text{ then } m_2 = m = 8$$

$$\text{If } m_2 = 2, \text{ then } m_1 = m = 1/2.$$

30. (1), (2), (3), (4)

The equation of any line making an acute angle θ with the positive direction of the x -axis and passing through $P(3, 4)$ is

$$\frac{x-3}{\cos \theta} = \frac{y-4}{\sin \theta} = r \quad (i)$$

where $|r|$ is the distance of any point (x, y) from P . Therefore, $A(r \cos \theta + 3, r \sin \theta + 4)$ is a general point on line (i). If A is R , then

$$r \cos \theta + 3 = 6 \text{ or } r = \frac{3}{\cos \theta} = 3 \sec \theta$$

Since θ is acute, $\cos \theta > 0$. Therefore,

$$PR = r = 3 \sec \theta$$

If $A \equiv S$, $r \sin \theta + 4 = 8$. Therefore,

$$r = 4 \operatorname{cosec} \theta$$

$$\therefore PS = 4 \operatorname{cosec} \theta$$

$$\text{Also, } PR + PS = \frac{3}{\cos \theta} + \frac{4}{\sin \theta}$$

$$= \frac{2(3 \sin \theta + 4 \cos \theta)}{\sin 2\theta}$$

$$\text{and } \frac{9}{(PR)^2} + \frac{16}{(PS)^2} = \cos^2 \theta + \sin^2 \theta = 1$$

Therefore, (1), (2), (3), and (4) all are correct.

Linked Comprehension Type

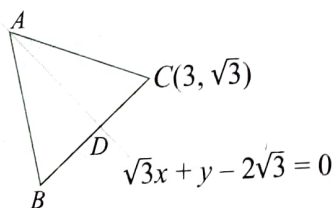
1. (1) Given lines $(x + y + 1) + b(2x - 3y - 8) = 0$ are concurrent at the point of intersection of the lines $x + y + 1 = 0$ and $2x - 3y - 8 = 0$, which is $(1, -2)$. Now, the line through $A(1, -2)$, which is farthest from the point $B(2, 2)$, is perpendicular to AB . Now, the slope of AB is 4. Then the required line is $y + 2 = -(1/4)(x - 1)$ or $x + 4y + 7 = 0$.

2. (3) Also, this line $x + 4y + 7 = 0$ meets the axes at $C(-7, 0)$ and $D(0, -7/4)$. Then the area of triangle OCD is $(1/2) |(-7)(-7/4)|$ or $49/8$.

3. (4) Lines $x + 4y + 7 = 0$ and $x - 2y + 1 = 0$ intersect at $(-3, -1)$ which must satisfy the line $3x - 4y + \lambda = 0$. Then $-9 + 4 + \lambda = 0$ or $\lambda = 5$.

4. (2), 5. (4), 6. (1)

Let the triangle be ABC with $C \equiv (3, \sqrt{3})$ and the altitude drawn through the vertex (meeting BC at D) be $\sqrt{3}x + y - 2\sqrt{3} = 0$. If B is (x_b, y_b) , then we have



$$\frac{x_b - 3}{\sqrt{3}} = \frac{y_b - \sqrt{3}}{1} = \frac{2(3\sqrt{3} + \sqrt{3} - 2\sqrt{3})}{4} = -\sqrt{3}$$

or $x_b = 0, y_b = 0$ and the coordinates of D are $(3/2, \sqrt{3}/2)$. Let the coordinates of vertex A be (x_a, y_a) . Then,

$$\frac{x_a - (3/2)}{-1/2} = \frac{y_a - (\sqrt{3}/2)}{\sqrt{3}/2} = \pm 3$$

or $(x_a, y_a) \equiv (0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

Hence, the remaining vertices are $(0, 0)$ and $(0, 2\sqrt{3})$ or $(0, 0)$ and $(3, -\sqrt{3})$. Also, the orthocenter is $(1, \sqrt{3})$ or $(2, 0)$.

7. (4) Let the parametric equation of the line drawn be

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta} = r$$

$$\text{or } x = r \cos \theta, y = r \sin \theta$$

Putting it in L_1 , we get

$$r \sin \theta = r \cos \theta + 10$$

$$\text{or } \frac{1}{OA} = \frac{\sin \theta - \cos \theta}{10}$$

Similarly, putting the general point of drawn line in the equation of L_2 , we get

$$\frac{1}{OB} = \frac{\sin \theta - \cos \theta}{20}$$

Let $P \equiv (h, k)$ and $OP = r$. Then, $r \cos \theta = h, r \sin \theta = k$. We have

$$\frac{2}{r} = \frac{\sin \theta - \cos \theta}{10} + \frac{\sin \theta - \cos \theta}{20}$$

$$\text{or } 40 = 3r \sin \theta - 3r \cos \theta$$

$$\text{or } 3y - 3x = 40$$

$$8. (3) r^2 = \frac{10 \times 20}{(\sin \theta - \cos \theta)^2}$$

$$\text{or } (r \sin \theta - r \cos \theta)^2 = 200$$

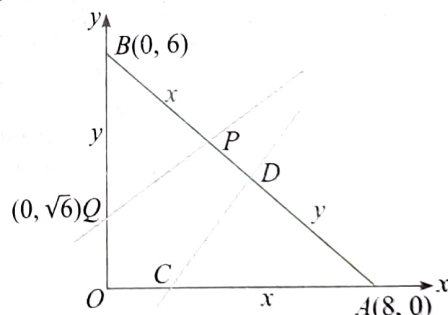
Hence, the locus is $(y - x)^2 = 200$.

$$9. (1) \frac{1}{r^2} = \frac{(\sin \theta - \cos \theta)^2}{100} + \frac{(\sin \theta - \cos \theta)^2}{400}$$

$$\text{or } 400 = 5(r \sin \theta - r \cos \theta)^2$$

Hence, the locus is $400 = 5(x - y)^2$, i.e., $(x - y)^2 = 80$.

10. (1), 11. (2), 12. (3)



Case I: Let the line L cuts the AO and AB at distances x and y , respectively, from A . Then, the area of the triangle with sides x and y is

$$\frac{1}{2} xy \sin(\angle CAD) = \frac{1}{2} \cdot xy \cdot \frac{3}{5} = \frac{3xy}{10} = 12$$

$$\text{or } xy = 40$$

Also, $x + y = 12$ (from perimeter bisection). Then x and y are the roots of $r^2 - 12r + 40 = 0$ which has imaginary roots.

Case II: If the line L cuts OB and BA at distances y and x , respectively, from B , then we have $xy = 30$ and $x + y = 12$.

Solving, we get $x = 6 + \sqrt{6}$ and $y = 6 - \sqrt{6}$.

Case III: If the line L cuts the sides OA and OB at distances x and y , respectively, from O , then

$$x + y = 12 \text{ and } xy = 24$$

$$\therefore x, y = 6 \pm 2\sqrt{3} \text{ (not possible)}$$

So there is a unique line possible. Let point P be (α, β) .

Using the parametric equation of AB , we have

$$\beta = 6 - \frac{3}{5}(6 + \sqrt{6})$$

and $\alpha = \frac{4}{5}(6 + \sqrt{6})$

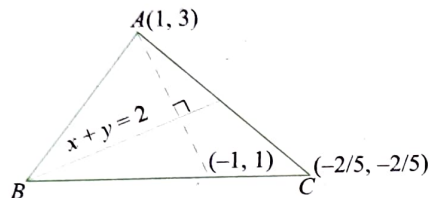
Hence, the slope of PQ is

$$\frac{\beta - \sqrt{6}}{\alpha - 0} = \frac{10 - 5\sqrt{6}}{10}$$

13. (2) The image of $A(1, 3)$ in line $x + y = 2$ is

$$(1 - 2(2)/2, 3 - 2(2)/2) \equiv (-1, 1)$$

So, line BC passes through $(-1, 1)$ and $(-2/5, -2/5)$. The equation of line BC is



$$y - 1 = \frac{-2/5 - 1}{-2/5 + 1}(x + 1)$$

or $7x + 3y + 4 = 0$

14. (3) Vertex B is the point of intersection of $7x + 3y + 4 = 0$ and $x + y = 2$, i.e., $B \equiv (-5/2, 9/2)$.

15. (1) Line AB is $y - 3 = \frac{3 - 9/2}{1 + 5/2}(x - 1)$
or $3x + 7y = 24$

16. (3) Angle between the diagonals is given by

$$\tan \theta = \left| \frac{-\frac{1}{2} + 2}{1 + 1} \right| = \frac{3}{4}$$

or $\sin \theta = \frac{3}{5}$

Area of $\triangle CPB$ is

$$\frac{1}{2} \times PC \times PB \sin \theta = 2$$

or $PB = \frac{10}{3}$

or $BD = \frac{20}{3}$

17. (1) $\cos(\pi - \theta) = \frac{AP^2 + PB^2 - AB^2}{2AP \times PB}$

or $-\frac{4}{5} = \frac{4 + (100/9) - AB^2}{2 \times 2 \times (10/3)}$

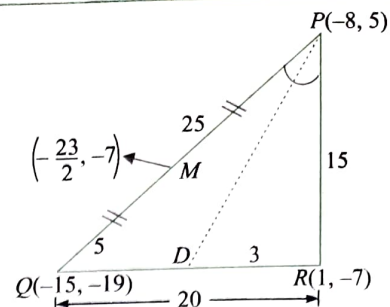
or $AB = \frac{2\sqrt{58}}{3}$

18. (1) In $\triangle CPB$,

$$\cos \theta = \frac{PC^2 + PB^2 - BC^2}{2PC \times PB}$$

or $BC = \frac{2\sqrt{10}}{3}$

19. (1) Since the triangle is right-angled, the circumcenter is the midpoint of PQ and the orthocenter is $R(1, -7)$. Hence,



$$RM = \left| \sqrt{\left(\frac{23}{2} + 1\right)^2} \right| = 12\frac{1}{2}$$

20. (2) The coordinates of the incentre are given as follows:

$$\bar{x} = \frac{20(-8) + 15(-15) + 25(1)}{20 + 15 + 25} = -6$$

$$\bar{y} = \frac{20(5) + 15(-19) + 25(-7)}{60} = -6$$

Hence, incentre I is $(-6, -6)$.

Now, the equation of side PR is

$$y + 7 = \frac{12}{-9}(x - 1)$$

or $4x - 4 = -3y - 21$

or $4x + 3y + 17 = 0$

Inradius is given as the distance of I from side PR , i.e.,

$$\frac{|-24 - 18 + 17|}{5} = 5$$

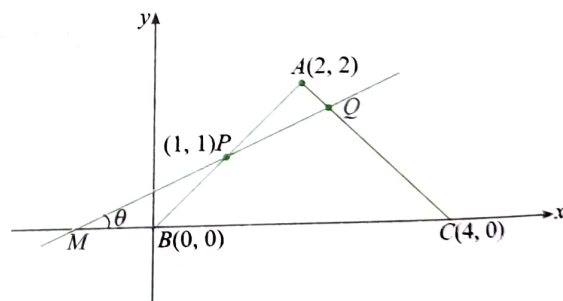
21. (3) The coordinates of D using section formula are $(-5, -23/2)$ and $m_{PD} = -11/2$. The equation of PD is

$$11x + 2y + 78 = 0$$

or $a + c = 89$

22. (2), 23. (4), 24. (3)

For the given specifications of the triangle, consider triangle ABC as shown in the figure.



Equation of line PM is

$$y - 1 = \tan \theta (x - 1)$$

For point Q , solving MP and AC ,

$$4 - x - 1 = \tan \theta (x - 1)$$

$$\Rightarrow Q \equiv \left(\frac{3 + \tan \theta}{1 + \tan \theta}, \frac{1 + 3 \tan \theta}{1 + \tan \theta} \right)$$

$$\text{Area of } \triangle APQ = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ \frac{3 + \tan \theta}{1 + \tan \theta} & \frac{1 + 3 \tan \theta}{1 + \tan \theta} & \frac{1 + 3 \tan \theta}{1 + \tan \theta} \end{vmatrix} = \frac{1 - \tan \theta}{1 + \tan \theta} \quad (\text{As } \theta < \pi/4)$$

Area of quadrilateral $BPOC$,

$$\Delta = \frac{1}{2} \times 4 \times 2 - \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right)$$

$$\Delta = \frac{3 + 5 \tan \theta}{1 + \tan \theta}$$

$$\Delta = 5 - \frac{2}{1 + \tan \theta}, \text{ where } \theta \in \left(0, \frac{\pi}{4} \right)$$

$$\Delta \in (3, 4)$$

$$(PQ)^2 = \left(\frac{3 + \tan \theta}{1 + \tan \theta} - 1 \right)^2 + \left(\frac{1 + 3 \tan \theta}{1 + \tan \theta} - 1 \right)^2$$

$$= \frac{4(1 + \tan^2 \theta)}{1 + \tan^2 \theta + 2 \tan \theta}$$

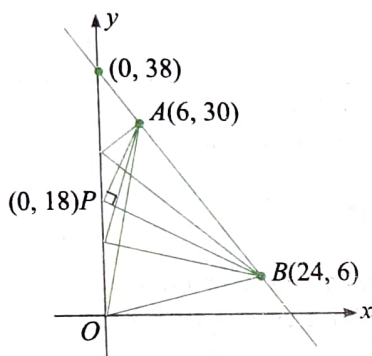
$$= \frac{4}{1 + \frac{2 \tan \theta}{1 + \tan^2 \theta}}$$

$$= \frac{4}{1 + \sin 2\theta}$$

So, $(PQ)^2 \in (2, 4)$

$\therefore PQ \in (\sqrt{2}, 2)$

35. (3), 26. (2), 27. (2)



Slope of AP is $m_1 = \frac{30 - \lambda}{6}$

Slope of BP is $m_2 = \frac{6 - \lambda}{24}$

$$\tan \theta = \frac{\frac{30 - \lambda}{6} - \frac{6 - \lambda}{24}}{1 + \left(\frac{30 - \lambda}{6} \right) \left(\frac{6 - \lambda}{24} \right)}$$

$$= \frac{720 - 24\lambda - 36 + 6\lambda}{144 + 180 - 36\lambda + \lambda^2}$$

$$= \frac{18(38 - \lambda)}{324 - 36\lambda + \lambda^2} = \frac{(38 - \lambda)18}{(\lambda - 18)^2}$$

Clearly, for $0 < \lambda < 38$,

Maximum value of $\tan \theta \rightarrow \infty$ for $\lambda = 18$, for which $\theta = \frac{\pi}{2}$.

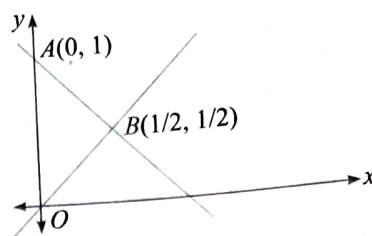
Now, $\frac{d}{d\lambda} (\tan \theta) = \frac{18(18 - \lambda)(\lambda - 58)}{(\lambda - 18)^4}$

Clearly, $\lambda = 58$ is point of maxima as derivative changes sign from '+' to '-'.

So, point $Q \equiv (0, 58)$ when $\angle AQB$ is maximum.

Matrix Match Type

1. $a \rightarrow s$; $b \rightarrow r$; $c \rightarrow p$; $d \rightarrow q$



The given lines are $x(x + y - 1)(x - y) = 0$. So, lines $x = 0$, $x + y - 1 = 0$, and $x - y = 0$ form triangle OAB as shown in the diagram.

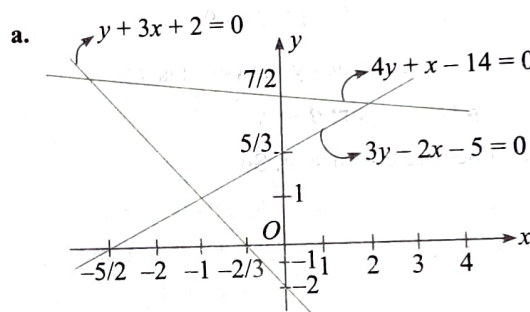
The triangle is right-angled at point B . Hence, the orthocenter is $(1/2, 1/2)$. Also, the circumcenter is the midpoint of OA which is $(0, 1/2)$. The centroid is

$$\left(\frac{0 + (1/2) + 0}{3}, \frac{0 + (1/2) + 1}{3} \right) \text{ or } \left(\frac{1}{6}, \frac{1}{2} \right)$$

Also, $OA = 1$, $OB = OC = 1/\sqrt{2}$. Hence, the incenter is

$$\left(\frac{0(1/\sqrt{2}) + (1/2)1 + 0(1/\sqrt{2})}{(1/\sqrt{2}) + 1 + (1/\sqrt{2})}, \frac{0(1/\sqrt{2}) + (1/2)(1) + 1(1/\sqrt{2})}{(1/\sqrt{2}) + 1 + (1/\sqrt{2})} \right) = \left(\frac{1}{2 + 2\sqrt{2}}, \frac{1}{2} \right)$$

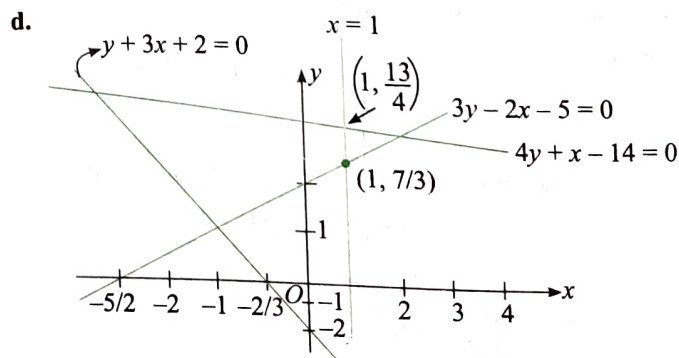
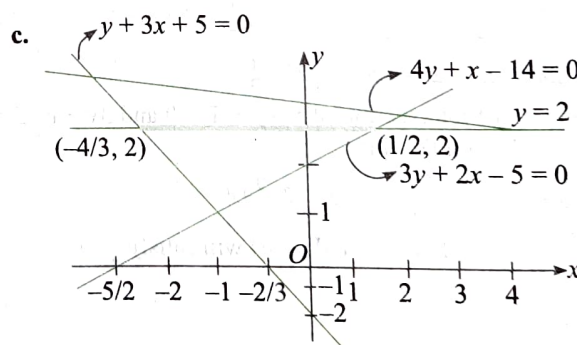
2. $a \rightarrow s$; $b \rightarrow r$; $c \rightarrow q$; $d \rightarrow p$



Clearly, point $(0, \alpha)$ lies on y -axis.

So, $5/3 < \alpha < 7/2$

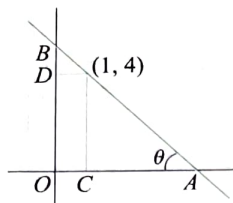
b. Clearly, point $(\alpha, 0)$ lies on the x -axis, which is not intersecting any side of triangle. Hence, no such α exists.



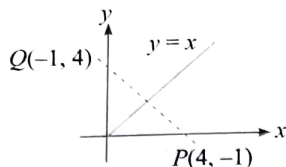
3. $a \rightarrow s$; $b \rightarrow p$; $c \rightarrow q$; $d \rightarrow r$

a.

$$\begin{aligned} OA &= 1 + 4 \cot \theta \\ OB &= 4 + \tan \theta \\ OA + OB &= 5 + 4 \cot \theta + \tan \theta \\ &\geq 5 + 2\sqrt{4 \cot \theta \tan \theta} \\ &= 5 + (2 \times 2) = 9 \end{aligned}$$



b.



The reflection of $P(4, -1)$ on $y = x$ is $Q(-1, 4)$. Hence,

$$PQ = \sqrt{(4+1)^2 + (-1-4)^2} = \sqrt{50} = 5\sqrt{2}$$

c.

$$AB = 2\sqrt{2}$$

$$OC = \sqrt{2}$$

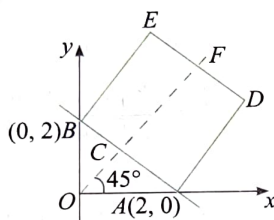
The maximum value of d is

$$\begin{aligned} OF &= \sqrt{2} + 2\sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$$

d. The given line is

$$x = 4 + \frac{1}{\sqrt{2}} \left(\frac{y+1}{\sqrt{2}} \right) \text{ or } y = 2x - 9$$

Hence, the intercept made by the x -axis is $9/2$.



4. $a \rightarrow p$; s ; $b \rightarrow q$; s ; $c \rightarrow p$; r ; $d \rightarrow s$

a. The given lines are concurrent. So,

$$\begin{vmatrix} 3 & 1 & -4 \\ 1 & -2 & -6 \\ \lambda & 4 & \lambda^2 \end{vmatrix} = 0$$

$$\text{or } \lambda^2 + 2\lambda - 8 = 0$$

$$\text{or } \lambda = 2, -4$$

b. The points are collinear. Hence,

$$\begin{vmatrix} \lambda+1 & 1 & 1 \\ 2\lambda+1 & 3 & 1 \\ 2\lambda+2 & 2\lambda & 1 \end{vmatrix} = 0$$

$$\text{or } 2\lambda^2 - 3\lambda - 2 = 0 \text{ or } \lambda = 2, -\frac{1}{2}$$

c. The point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ is $(1, 2)$. It lies on the line

$$x + y - 1 - \frac{\lambda}{2} = 0 \quad \therefore \lambda = \pm 4$$

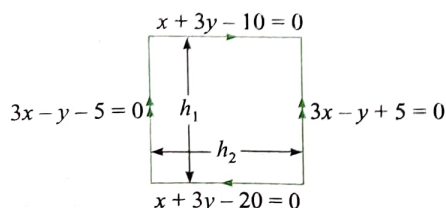
d. The midpoint of $(1, -2)$ and $(3, 4)$ will satisfy

$$y - x - 1 + \lambda = 0$$

$$\therefore \lambda = 2$$

5. $a \rightarrow q$; r ; s ; $b \rightarrow p$; $c \rightarrow q$; s ; $d \rightarrow q$

a.



$$h_1 = \left| \frac{10}{\sqrt{10}} \right| = \sqrt{10}$$

$$h_2 = \frac{10}{\sqrt{10}} = \sqrt{10}$$

Hence, the given lines form a square of side $\sqrt{10}$. Therefore, the area is 10 sq. units.

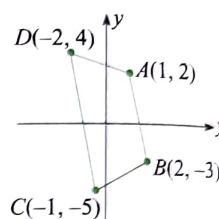
$$b. m_{AB} = \frac{5}{-1} = -5$$

$$m_{DC} = \frac{9}{-1} = -9$$

Hence, the figure is not a parallelogram.

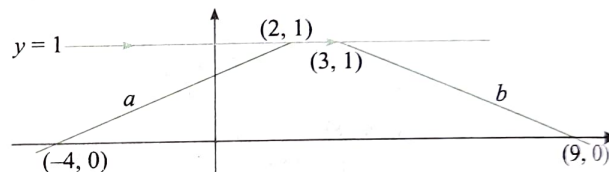
c. Lines $7x + 3y - 33 = 0$ and $7x + 3y - 4 = 0$ are parallel. The distance between them is $|29/\sqrt{58}|$. Lines $3x - 7y + 19 = 0$ and $3x - 7y - 10 = 0$ are parallel. The distance between them is $|29/\sqrt{58}|$. Also, lines $7x + 3y - 33 = 0$ and $3x - 7y + 19 = 0$ are perpendicular. Hence, the given lines form a square.

d. Lines $4y - 3x - 7 = 0$ and $4y - 3x - 21 = 0$ are parallel. Lines $3y - 4x + 7 = 0$ and $3y - 4x + 14 = 0$ are parallel. Also, lines $4y - 3x - 7 = 0$ and $3y - 4x + 14 = 0$ are not perpendicular. Hence, the given lines form a parallelogram.



6. $a \rightarrow p$; s ; $b \rightarrow p$; $c \rightarrow q$; $d \rightarrow p$; q ; r .

a.



Obviously, it is a trapezium.

$$a = \sqrt{37}$$

$$b = \sqrt{37}$$

$$\therefore a = b$$

Hence, it is an isosceles trapezium.

Therefore, it is a cyclic quadrilateral.

b. $ac = bd$

$$\text{or } \frac{b}{c} = \frac{a}{d}$$

$$\tan \theta = \frac{b}{c}$$

$$\tan \phi = \frac{a}{d}$$

$$\text{or } \theta = \phi$$

Hence, it is a cyclic quadrilateral.

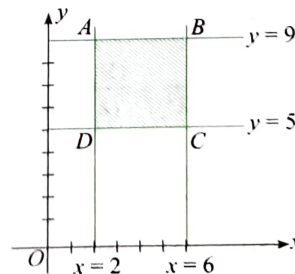
c. $ax \pm by \pm c = 0$

$$\text{If } y = 0, y = \pm \frac{c}{a}$$

$$\text{If } x = 0, y = \pm \frac{c}{b}$$

Therefore, it is a rhombus.

d.



$$(x - 6)(x - 2) = 0$$

$$x = 6 \text{ and } x = 2$$

$$y^2 - 14y + 45 = 0$$

$$(y - 9)(y - 5) = 0$$

Therefore, it is a square.

1. a $\rightarrow$ s; b $\rightarrow$ p, q; c $\rightarrow$ r; d $\rightarrow$ p, q, s

Given lines are

$$L_1: x + 3y - 5 = 0$$

$$L_2: 3x - ky - 1 = 0$$

$$L_3: 5x + 2y - 12 = 0$$

L_1 and L_3 intersect at (2, 1)

$\therefore L_1, L_2, L_3$ are concurrent if

$$6 - k - 1 = 0 \text{ or } k = 5$$

For L_1, L_2 to be parallel

$$\frac{1}{3} = \frac{3}{-k} \Rightarrow k = -9$$

For L_2, L_3 to be parallel

$$\frac{3}{5} = \frac{-k}{2} \Rightarrow k = \frac{-6}{5}$$

Thus, for $k = 5$, lines are concurrent and for $k = -9, \frac{-6}{5}$, at least two lines are parallel. So, for these values of k , lines will not form triangle.

Obviously, for $k \neq 5, -9, \frac{-6}{5}$, lines form triangle.

8. (3)

a. The perpendicular bisector of AB is parallel to $x - y - 4 = 0$. Therefore, the slope is 1.

It is passing through the circumcenter, i.e., $(3/2, 5/2)$.

Therefore, the line is $x - y + 1 = 0$.

b. The perpendicular bisector of AC is B parallel to $2x - y - 5 = 0$.

Therefore, the slope is 2.

It is passing through the circumcenter, i.e., $(3/2, 5/2)$.

Therefore, the line is $4x - 2y - 1 = 0$.

c. The equation of perpendicular bisector to AC is $2x - y - 5 = 0$.

Therefore, the equation of AC can be $x + 2y + \lambda = 0$.

It is passing through $A(-2, 3)$. Therefore,

$$\lambda = -4$$

Therefore, the equation of AC is $x + 2y - 4 = 0$.

d. The image of A on the perpendicular bisector of AC is

$$\frac{x+2}{4} = \frac{y-3}{-2} = \frac{-2(-8-6-1)}{16+4} = \frac{3}{2}$$

$$\therefore C \equiv (4, 0)$$

The image of A on the perpendicular bisector of AB is

$$\frac{x+2}{1} = \frac{y-3}{-1} = \frac{-2(-2-3+1)}{1+1} = 4$$

Therefore, B is (2, -1).

Therefore, the midpoint of BC is $D(3, -1/2)$.

Hence, the median through A is $7x + 10y - 16 = 0$.

Numerical Value Type

1. (18) Let the equation of line be $(y - 2) = m(x - 8)$ where $m < 0$.

$$\Rightarrow P \equiv \left(8 - \frac{2}{m}, 0\right) \text{ and } Q \equiv (0, 2 - 8m)$$

$$\text{Now, } OP + OQ = \left|8 - \frac{2}{m}\right| + |2 - 8m|$$

$$= 10 + \frac{2}{-m} + (-8m)$$

$$\geq 10 + 2\sqrt{\frac{2}{-m} \times (-8m)} \geq 18$$

2. (1) Lines $(k+1)x + 8y = 4k$ and $kx + (k+3)y = 3k - 1$ are coincident. Then we can compare the ratio of coefficients as

$$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1}$$

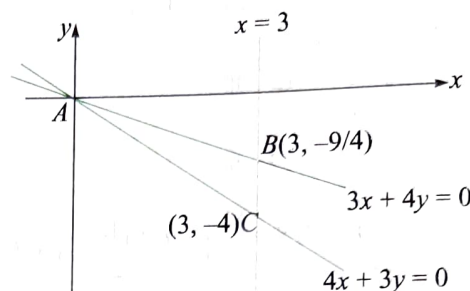
$$\text{or } k^2 + 4k + 3 = 8k \text{ and } 24k - 8 = 4k^2 + 12k$$

$$\text{or } (k-3)(k-1) = 0 \text{ and } (k-2)(k-1) = 0$$

$$\text{or } k = 1$$

3. (0) The equation of the angle bisector of angle A is

$$\frac{3x+4y}{5} = \pm \frac{4x+3y}{5} \text{ or } x = \pm y$$



The equation of internal bisector is $x = -y$

Since h and k lie on the line $x = -y$, we have

$$h + k = 0$$

4. (4) Any point on the line $x + y = 4$ is $(t, 4 - t)$, where $t \in R$.

Now, the distance of this point from the line $4x + 3y - 10 = 0$ is 1. Therefore,

$$\frac{|4t + 3(4 - t) - 10|}{5} = 1$$

$$\text{or } |t + 2| = 5$$

$$\text{i.e., } t = 3 \text{ or } t = -7$$

Therefore, the sum of values is -4

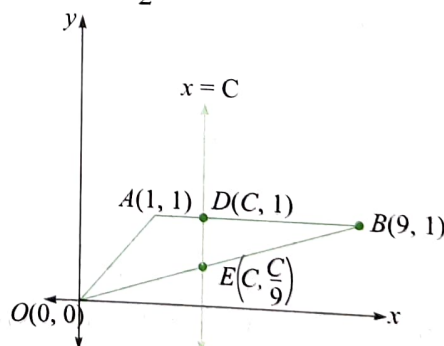
5. (3) Distance of $(0, 0)$ from the given lines is

$$a = \frac{|0 + 0 + 5|}{5} = 1$$

$$b = \frac{|0 - 0 + 15|}{5} = 3$$

$$\therefore \text{Area of rectangle} = ab = 3 \text{ sq. units}$$

6. (3) Area of $\triangle OAB = \frac{1}{2} (1) (8) = 4 \text{ sq. units}$



The equation of OB is

$$y = \frac{1}{9}x$$

Hence, the point E is $(C, C/9)$.

Now, the area of $\triangle BDE$ is 2 sq. units. Therefore,

$$\text{or } \frac{1}{2} \left(1 - \frac{C}{9}\right) (9 - C) = 2$$

$$\text{or } (9 - C)^2 = 36$$

$$\text{or } 9 - C = \pm 6$$

$$\text{or } C = 3$$

7. (-2) Lines $(2a + b)x + (a + 3b)y + (b - 3a) = 0$ or $a(2x + y - 3) + b(x + 3y + 1) = 0$ are concurrent at the point of intersection of lines $2x + y - 3 = 0$ and $x + 3y + 1 = 0$, which is $(2, -1)$.
Now, line $mx + 2y + 6 = 0$ must pass through this point. Therefore, $2m - 2 + 6 = 0$ or $m = -2$.

8. (91) Line $3x + 2y = 24$ meets the axes at $B(8, 0)$ and $A(0, 12)$. The midpoint of AB is $D(4, 6)$.

The equation of perpendicular bisector of AB is

$$2x - 3y + 10 = 0 \quad (1)$$

Now, the line through $(0, -1)$ and parallel to the x -axis is $y = -1$.

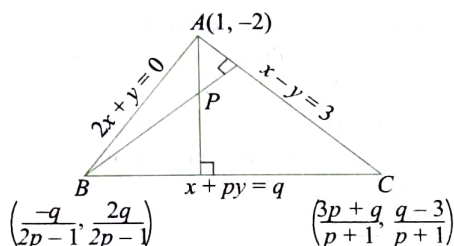
The coordinates of C where line (1) meets $y = -1$ are $(-13/2, -1)$.

Now, the area of triangle ABC ,

$$\Delta = \frac{1}{2} \begin{vmatrix} 0 & 12 & 1 \\ 8 & 0 & 1 \\ -13/2 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[0 - 12 \left(8 + \frac{13}{2} \right) + 1(-8) \right] = \frac{1}{2} [-6(29) - 8] = 91$$

9. (50)



P is the orthocenter. Therefore,

$$AP \perp BC$$

$$\text{or } \left(-\frac{1}{p} \right) \left(\frac{3+2}{2-1} \right) = -1$$

$$\text{or } \frac{5}{p} = 1 \text{ or } p = 5$$

Since $BP \perp AC$, we have

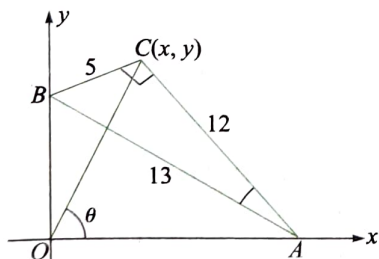
$$\frac{27-2q}{18+q} = -1$$

$$\text{or } q = 27 + 18$$

$$\text{or } q = 45$$

$$\therefore p + q = 5 + 45 = 50$$

10. (5)



Since $\angle BCA = 90^\circ$, points A, O, B, C are concyclic.

Let $\angle AOC = \theta$.

$$\angle BOC = \angle BAC$$

$$\tan \left(\frac{\pi}{2} - \theta \right) = \frac{5}{12}$$

$$\frac{x}{y} = \frac{5}{12} \text{ or } 12x - 5y = 0$$

11. (3) Point P which lies on the line $y = 3x/4$ can be chosen as $P(h, 3h/4)$.

If θ is the angle that the line $y = 3x/4$ makes with the positive direction of the x -axis, then

$$\tan \theta = \frac{3}{4} \text{ or } \cos \theta = \frac{4}{5} \text{ and } \sin \theta = \frac{3}{5}$$

Now, the coordinates of points A and B which lie on the line $y = 3x/4$ can be chosen as

$$A \equiv \left(h + \frac{4r_1}{5}, \frac{3h}{4} + \frac{3r_1}{5} \right) \text{ and } B \equiv \left(h + \frac{4r_2}{5}, \frac{3h}{4} + \frac{3r_2}{5} \right)$$

Since A lies on the line $x - y + 1 = 0$, we have

$$\left(h + \frac{4r_1}{5} \right) - \left(\frac{3h}{4} + \frac{3r_1}{5} \right) + 1 = 0$$

$$\text{or } r_1 = \frac{-5}{4} (h + 4)$$

Since B lies on the line $2x - y - 5 = 0$, we have

$$2 \left(h + \frac{4r_2}{5} \right) - \left(\frac{3h}{4} + \frac{3r_2}{5} \right) - 5 = 0$$

$$\text{or } r_2 = \frac{-5}{4} (h - 4)$$

According to the given condition, we have

$$PA \cdot PB = 25$$

$$\text{i.e., } |r_1| \cdot |r_2| = 25$$

$$\text{i.e., } \frac{25}{16} (h^2 - 16) = \pm 25$$

$$\text{i.e., } h^2 = 16 \pm 16 = 32, 0$$

$$\text{i.e., } h = \pm 4\sqrt{2}, 0$$

Hence, the required points are $(0, 0)$, $(4\sqrt{2}, 3\sqrt{2})$, and $(-4\sqrt{2}, -3\sqrt{2})$.

12. (9) Each family has parallel lines having the distance between them as $1/\sqrt{2}$ unit. Both the families are perpendicular to each other. So, to form a square of diagonal 2 units, lines of alternate pair are to be chosen.

Both the families have three such pairs. So, the number of squares possible is $3 \times 3 = 9$.

13. (20) Equation of line $\frac{ax}{c-1} + \frac{by}{c-1} + 1 = 0$ has two independent parameters.

$$5a + 5b + 20c = t$$

$$\therefore 5a + 5b = t - 20c$$

$$\therefore \frac{5a}{c-1} + \frac{5b}{c-1} = \frac{t-20c}{c-1}$$

R.H.S. is independent of c if $t = 20$.

Archives

JEE Main

Single Correct Answer Type

1. (1) The slope of line L is $-b/5$.

The slope of line K is $-3/c$.

The line L is parallel to the line K . Therefore,

$$\frac{b}{5} = \frac{3}{c} \Rightarrow bc = 15$$

$(13, 32)$ is a point on L . So

$$\frac{13}{5} + \frac{32}{b} = 1$$

$$\Rightarrow \frac{32}{b} = -\frac{8}{5} \Rightarrow b = -20$$

$$\therefore c = -\frac{3}{4}$$

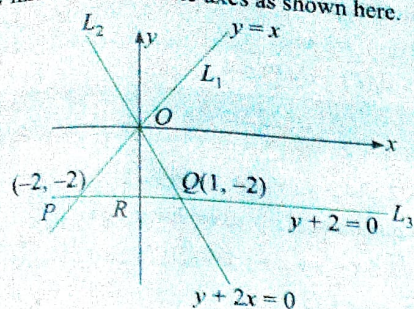
The equation of K is $y - 4x = 3$.

Hence, the distance between L and K

$$= \text{distance of point } (13, 32) \text{ from line } K$$

$$= \frac{|52 - 32 + 3|}{\sqrt{17}} = \frac{23}{\sqrt{17}}$$

1. (i) Draw lines on coordinate axes as shown here.



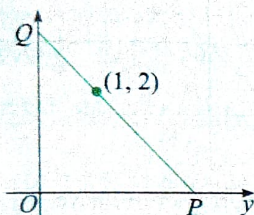
From the geometry of angle bisector,

$$\frac{PR}{RQ} = \frac{OP}{OQ} = \frac{2\sqrt{2}}{\sqrt{5}}$$

$$(3) (y-2) = m(x-1)$$

$$\therefore OP = 1 - \frac{2}{m}$$

$$OQ = 2 - m$$



Area of ΔPOQ

$$= \frac{1}{2} (OP) (OQ) = \frac{1}{2} \left(1 - \frac{2}{m}\right) (2 - m)$$

$$= \frac{1}{2} \left[2 - m - \frac{4}{m} + 2\right]$$

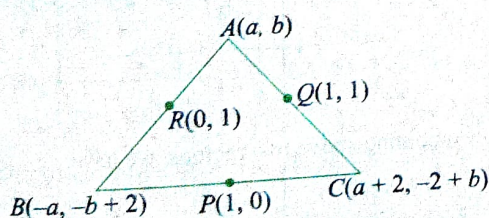
$$= \frac{1}{2} \left[4 - \left(m + \frac{4}{m}\right)\right]$$

$$= \frac{1}{2} \left[\left(\sqrt{-m} - \frac{2}{\sqrt{-m}}\right)^2 + 8 \right]$$

$$(\because m < 0)$$

Which has least value when $m = -2$.

4. (2)



Let vertex A be (a, b) .

R is mid point of AB.

$$\therefore B \text{ is } (-a, -b+2)$$

P is mid point of BC.

$$\therefore C \text{ is } (a+2, -2+b)$$

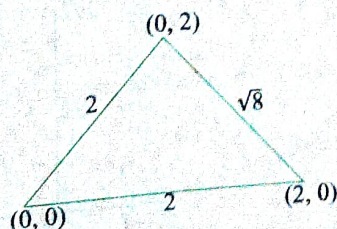
Q is mid point of AC.

$$\therefore a = 0 \text{ and } b = 2$$

$\therefore$ Abscissa of in center

$$I_x = \frac{0 \times 2 + 0 \times \sqrt{8} + 2 \times 2}{2 + 2 + 2\sqrt{2}}$$

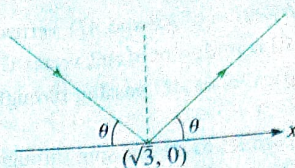
$$\therefore I_x = \frac{4}{4 + 2\sqrt{2}}$$



$$\Rightarrow I_x = \frac{2}{2 + \sqrt{2}} \times \frac{2 - \sqrt{2}}{2 - \sqrt{2}}$$

$$\Rightarrow I_x = \frac{2(2 - \sqrt{2})}{2} = 2 - \sqrt{2}$$

$$5. (2) \text{ Slope of } x + \sqrt{3}y = \sqrt{3} \text{ is } -\frac{1}{\sqrt{3}} = m_1 \text{ (let)}$$



$$\text{So, } \tan \theta = -\frac{1}{\sqrt{3}}$$

$$\theta = 150^\circ$$

$$\text{So, slope of reflected ray} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

So, equation of reflected ray is

$$y - 0 = \frac{1}{\sqrt{3}} (x - \sqrt{3})$$

$$\text{or } \sqrt{3}y = x - \sqrt{3}$$

6. (3) Since point of intersection lies in IV quadrant and equidistant from axes, let the point of intersection be $(h, -h)$, $h > 0$

$$\Rightarrow 4ah - 2ah + c = 0$$

$$\text{and } 5bh - 2bh + d = 0$$

$$\text{So, } -\frac{c}{2a} = -\frac{d}{3b}$$

$$\Rightarrow 3bc - 2ad = 0$$

7. (2) S is the midpoint of QR,

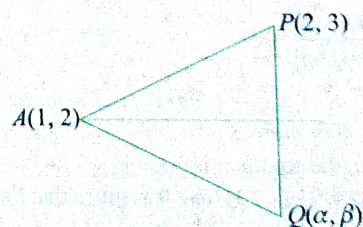
$$\therefore S\left(\frac{13}{2}, 1\right)$$

$$\text{Slope of PS} = -\frac{2}{9}$$

$$\therefore \text{Equation of required line is: } y + 1 = -\frac{2}{9}(x - 1)$$

$$\therefore 2x + 9y + 7 = 0$$

8. (3) Lines $(2x - 3y + 4) + k(x - 2y + 3) = 0$ pass through point of intersection of $2x - 3y + 4 = 0$ and $x - 2y + 3 = 0$ which is $A(1, 2)$.



Let image of P(2, 3) in any line through A be Q(α , β).

$$\therefore AP = AQ$$

$$\therefore (\alpha - 1)^2 + (\beta - 2)^2 = (2 - 1)^2 + (3 - 2)^2$$

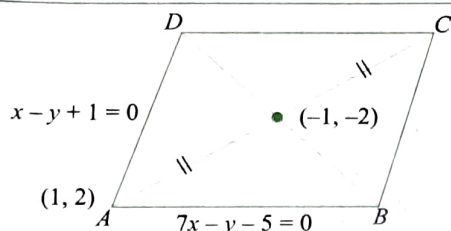
$$\therefore \alpha^2 + \beta^2 - 2\alpha - 4\beta + 3 = 0$$

$$\text{or } x^2 + y^2 - 2x - 4y + 3 = 0$$

which is a circle having radius,

$$r = \sqrt{1 + 4 - 3} = \sqrt{2}$$

9. (2)



On solving equation of AB and AD, vertex A(1, 2)

$\therefore (-1, -2)$ is midpoint of AC, vertex C is $(-3, -6)$.

Line BC is parallel to AD passing through point C, hence its equation is $x - y - 3 = 0$

Line DC is parallel to AB passing through point C, hence its equation is $7x - y + 15 = 0$

Hence other vertices are

$$B\left(\frac{1}{3}, -\frac{8}{3}\right) \text{ and } D\left(-\frac{7}{3}, -\frac{4}{3}\right)$$

JEE ADVANCED

Single Correct Answer Type

1. (4) The intersection point of $y = 0$ with the first line is $B(-p, 0)$.
The intersection point of $y = 0$ with the second line is $A(-q, 0)$.

The intersection point of the two lines is

$$C(pq, (p+1)(q+1))$$

The altitude from C to AB is $x = pq$.

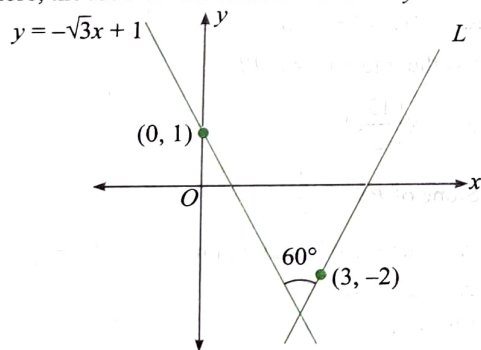
The altitude from B to AC is

$$y = -\frac{q}{1+q}(x+p)$$

Solving these two, we get $x = pq$ and $y = -pq$.

Therefore, the locus of the orthocenter is $x + y = 0$.

2. (2)



Let the slope of the required line be m . Then

$$\left| \frac{m + \sqrt{3}}{1 - \sqrt{3}m} \right| = \sqrt{3}$$

$$\therefore m + \sqrt{3} = \pm(\sqrt{3} - 3m)$$

$$\therefore m = 0 \text{ or } m = \sqrt{3}$$

Therefore, the equation is

$$y + 2 = \sqrt{3}(x - 3) \quad (m \neq 0 \text{ as given that line cuts the } x\text{-axis})$$

$$\text{or } \sqrt{3}x - y - (2 + 3\sqrt{3}) = 0$$

3. (1) Solving given lines for their point of intersection, we get the point of intersection as $(-c/(a+b), -c/(a+b))$.

Its distance from (1, 1) is

$$\sqrt{\left(1 + \frac{c}{a+b}\right)^2 + \left(1 + \frac{c}{a+b}\right)^2} < 2\sqrt{2} \quad (\text{given})$$

$$\text{or } (a+b+c)^2 < 4(a+b)^2 \text{ or } (a+b+c)^2 - (2a+2b)^2 < 0$$

$$\text{or } (c-a-b)(c+3a+3b) < 0$$

$$\text{Since } a > b > c > 0, (c-a-b) < 0 \text{ or } a+b-c > 0$$

Numerical Value Type

1. (6) For $P(x, y)$, we have

$$2 \leq d_1(P) + d_2(P) \leq 4$$

$$\Rightarrow 2 \leq \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \leq 4$$

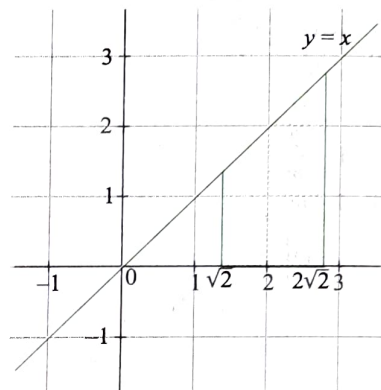
$$2\sqrt{2} \leq |x-y| + |x+y| \leq 4\sqrt{2}$$

In first quadrant, if $x > y$, we have

$$2\sqrt{2} \leq x - y + x + y \leq 4\sqrt{2}$$

$$\text{or } \sqrt{2} \leq x \leq 2\sqrt{2}$$

The region of points satisfying these inequalities is

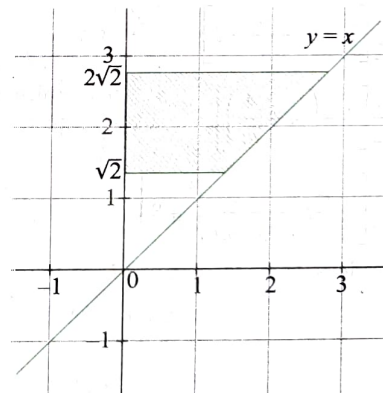


In first quadrant, if $x < y$, we have

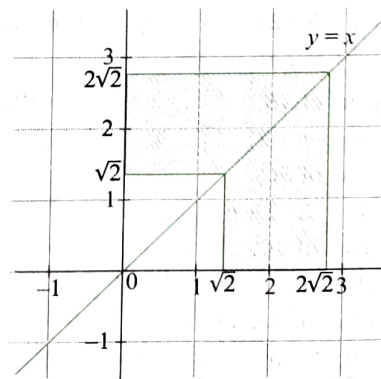
$$2\sqrt{2} \leq y - x + x + y \leq 4\sqrt{2}$$

$$\text{or } \sqrt{2} \leq y \leq 2\sqrt{2}$$

The region of points satisfying these inequalities is



Combining above two regions, we have



$$\begin{aligned} \text{Area of the shaded region} &= \left((2\sqrt{2})^2 - (\sqrt{2})^2 \right) \\ &= 8 - 2 = 6 \text{ sq. units} \end{aligned}$$

Concept Application Exercises

Exercise 3.1

1. The given pair of lines is $(2x + y)(x - y) = 0$.
So, the required pair of lines is
 $(2x + y + k)(x - y + k') = 0$
where $2x + y + k = 0$ and $x - y + k' = 0$ pass through
 $(1, 0)$. Therefore,
 $k = -2, k' = -1$

Therefore, the required pair of lines is

$$(2x + y - 2)(x - y - 1) = 0$$

$$\text{or } 2x^2 - y^2 - xy - 4x + y + 2 = 0$$

2. The given equation is

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$$

Writing (i) as a quadratic equation in x , we get

$$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$$

$$\begin{aligned} \text{or } x &= \frac{-(5y + 6) \pm \sqrt{(5y + 6)^2 - 4 \times 2(3y^2 + 7y + 4)}}{4} \\ &= \frac{-(5y + 6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4} \\ &= \frac{-(5y + 6) \pm \sqrt{y^2 + 4y + 4}}{4} = \frac{-(5y + 6) \pm (y + 2)}{4} \\ \therefore x &= \frac{-5y - 6 + y + 2}{4}, \frac{-5y - 6 - y - 2}{4} \end{aligned}$$

$$\text{or } 4x + 4y + 4 = 0 \text{ and } 4x + 6y + 8 = 0$$

$$\text{or } x + y + 1 = 0 \text{ and } 2x + 3y + 4 = 0$$

Hence, (1) represents a pair of straight lines whose equations are

$$x + y + 1 = 0 \quad (1)$$

$$\text{and } 2x + 3y + 4 = 0 \quad (2)$$

Solving these two equations, the required point of intersection is $(1, -2)$.

Angle between lines,

$$\tan \theta = \left| \frac{-1 - \left(-\frac{2}{3}\right)}{1 + (-1)\left(-\frac{2}{3}\right)} \right| = \frac{1}{5}. \text{ So, } \theta = \tan^{-1} \frac{1}{5}$$

3. The bisector of the angle between the positive directions of the axes is $y = x$.

Since it is one of the lines of the given pair of lines $ax^2 + 2hxy + by^2 = 0$, we have

$$x^2(a + 2h + b) = 0 \text{ or } a + b = -2h.$$

4. The given equation of pair of straight lines can be rewritten as

$$(\sqrt{3}x - y)(x - \sqrt{3}y) = 0$$

Their separate equations are

$$y = \sqrt{3}x \text{ and } y = \frac{1}{\sqrt{3}}x$$

$$\text{or } y = \tan 60^\circ x \text{ and } y = \tan 30^\circ x$$

After rotation, the separate equations are

$$y = \tan 90^\circ x \text{ and } y = \tan 60^\circ x$$

$$\text{or } x = 0 \text{ and } y = \sqrt{3}x$$

Therefore, the combined equation of lines in the new position is

$$x(\sqrt{3}x - y) = 0 \text{ or } \sqrt{3}x^2 - xy = 0$$

5. The given equation will represent a pair of real and distinct lines if

$$h^2 > ab$$

$$\text{i.e., } \left(-\frac{k}{2}\right)^2 > (2)(2)$$

$$\text{or } k^2 > 16$$

$$\text{or } (k - 4)(k + 4) > 0$$

$$\text{i.e., } k \in (-\infty, -4) \cup (4, \infty)$$

6. Let $\phi(x, y) = 6x^2 + 5xy - 21y^2 + 13x + 38y - 5 = 0$.

Differentiating with respect to x treating y as constant, we get

$$\frac{d\phi}{dx} = 12x + 5y + 13$$

Differentiating with respect to y treating x as constant, we get

$$\frac{d\phi}{dy} = 5x - 42y + 38$$

Solving equations $12x + 5y + 13 = 0$ and $5x - 42y + 38 = 0$, we get

$$x = -\frac{32}{23} \text{ and } y = \frac{17}{23}$$

Therefore, the point of intersection is $(-32/23, 17/23)$.

Exercise 3.2

1. If m and $2m$ are the slopes, then

$$m + 2m = -\frac{2/h}{1/b} = -\frac{2b}{h}$$

$$\text{and } m \times 2m = \frac{1/a}{1/b} = \frac{b}{a}$$

Eliminating m , we get

$$2\left(-\frac{2b}{3h}\right)^2 = \frac{b}{a}$$

$$\text{or } \frac{ab}{h^2} = \frac{9}{8}$$

2. Let ϕ be the angle between the lines represented by

$$x^2 + 2xy \sec \theta + y^2 = 0 \quad (1)$$

Here, $a = 1$, $b = 1$, and $h = \sec \theta$. Hence,

$$\begin{aligned} \tan \phi &= \frac{2\sqrt{\sec^2 \theta - 1} \times 1}{1 + 1} \\ &= \frac{2\sqrt{\sec^2 \theta - 1}}{2} = \tan \theta \end{aligned}$$

$$\therefore \phi = \theta$$

Hence, the angle between the lines is θ .

3. The equation of the line is

$$3x - 2y = 1 \quad (1)$$

and the equation of the curve is

$$3x^2 + 5xy - 3y^2 + 2x + 3y = 0 \quad (2)$$

Making (2) homogenous with the help of (1), we get

$$3x^2 + 5xy - 3y^2 + (2x + 3y)(3x - 2y) = 0$$

$$\text{or } 9x^2 + 10xy - 9y^2 = 0 \quad (3)$$

which is the equation of the lines joining the origin to the point of intersection of (1) and (2).

Here, $a = 9$, $b = -9$.

Since $a + b = 0$, the two lines given by (3) are at right angles.

4. Writing the given equation as quadratic in x , we have

$$6x^2 + (5y + 7)x - (4y^2 - 13y + 3) = 0$$

$$\text{or } x = \frac{-(5y + 7) \pm \sqrt{(5y + 7)^2 + 24(4y^2 - 13y + 3)}}{12}$$

$$= \frac{-(5y+7) \pm \sqrt{121y^2 - 242y + 121}}{12}$$

$$= \frac{-5(y+7) \pm 11(y-1)}{12} = \frac{6y-18}{12}, \frac{-16y+4}{12}$$

or $2x - y + 3 = 0$ and $3x + 4y - 1 = 0$

which are the two lines represented by the given equation. The point of intersection is $(-1, 1)$ that is obtained by solving these equations.

Also, $\tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b}$

$$= \frac{2\sqrt{(5/2)^2 - 6(-4)}}{6-4} = \frac{\sqrt{121}}{2} = \frac{11}{2}$$

($\because a = 6, b = -4, h = 5/2$)

So, the equation of the required line is

$$y - 1 = \frac{11}{2}(x + 1)$$

$$\text{or } 11x - 2y + 13 = 0$$

5. Equation of the given lines is $ax^2 + 2hxy + by^2 = 0$.

Equation of the pair of bisectors is

$$h(x^2 - y^2) = (a - b)xy$$

or $hx^2 - (a - b)xy - hy^2 = 0$ (1)

$\therefore A = h, B = -h, 2H = -(a - b)$

Equation of the pair of bisectors of (1) is

$$H(x^2 - y^2) = (A - B)xy$$

or $-\frac{(a - b)}{2}(x^2 - y^2) = 2hxy$

or $-(a - b)(x^2 - y^2) = 4hxy$

or $(a - b)(x^2 - y^2) + 4hxy = 0$

6. The equation of lines joining the origin to the points of intersection of the given line and curve is

$$5x^2 + 11xy - 8y^2 + (8x - 4y)\left(\frac{x - y}{2}\right) + 12\left(\frac{x - y}{2}\right)^2 = 0$$

or $12x^2 - xy - 3y^2 = 0$

The equation of bisectors is

$$\frac{x^2 - y^2}{12 - (-3)} = \frac{xy}{-1/2}$$

or $x^2 + 30xy - y^2 = 0$

7. The equation of the first pair of lines is

$$2x^2 + 6xy + y^2 = 0$$

The equation of the pair of bisectors is

$$3(x^2 - y^2) = (2 - 1)xy$$

or $3(x^2 - y^2) = xy$ (1)

The equation of the second pair of lines is $4x^2 + 18xy + y^2 = 0$.

The equation of the pair of bisectors is $9(x^2 - y^2) = (4 - 1)xy$

$$9(x^2 - y^2) = 3xy$$

or $3(x^2 - y^2) = xy$ (2)

Equations (1) and (2) are the same.

The given pairs have same angular bisectors.

2. (1) $3a + a^2 - 2 = 0$

or $a^2 + 3a - 2 = 0$

or $a = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2}$

Therefore, there are two values of a .

3. (1) $9x^2 - 24xy + 16y^2 - 12x + 16y - 12 = 0$

or $(3x - 4y + 2)(3x - 4y - 6) = 0$

Hence, the distance between the lines is $\frac{|6 - (-2)|}{5} = \frac{8}{5}$.

4. (1) Acute angle between the lines $x^2 + 4xy + y^2 = 0$ is $\tan^{-1}\{(2\sqrt{4-1})/(1+1)\} = \tan^{-1}\sqrt{3} = \pi/3$. The angle bisectors of $x^2 + 4xy + y^2 = 0$ are given by

$$\frac{x^2 - y^2}{1 - 1} = \frac{xy}{2}$$

or $x^2 - y^2 = 0$ or $x = \pm y$

As $x + y = 0$ is perpendicular to $x - y = 4$, the given triangle is isosceles with vertical angle equal to $\pi/3$. Hence, it is an equilateral triangle.

5. (2) The straight lines represented by $(y - mx)^2 = a^2(1 + m^2)$ are

$$y - mx = \pm a\sqrt{1 + m^2}$$

i.e., $y - mx = a\sqrt{1 + m^2}$ (1)

and $y - mx = -a\sqrt{1 + m^2}$ (2)

Similarly, the straight lines represented by $(y - nx)^2 = a^2(1 + n^2)$ are

$$y - nx = a\sqrt{1 + n^2}$$
 (3)

and $y - nx = -a\sqrt{1 + n^2}$ (4)

Since the lines (1) and (2) are parallel, the distance between them is

$$\left| \frac{a\sqrt{1 + m^2} + a\sqrt{1 + m^2}}{\sqrt{1 + m^2}} \right| = |2a|$$

Similarly, the lines (3) and (4) are parallel lines and the distance between them is $|2a|$. Since the distances between parallel lines are the same, the four lines form a rhombus.

6. (2) Let $y = mx$ be a line common to the given pairs of lines. Then,

$$am^2 + 2m + 1 = 0 \text{ and } m^2 + 2m + a = 0$$

$$\therefore \frac{m^2}{2(1-a)} = \frac{m}{a^2-1} = \frac{1}{2(1-a)}$$

or $m^2 = 1$ and $m = -\frac{a+1}{2}$

or $(a+1)^2 = 4$

or $a = 1$ or -3

But for $a = 1$, the two pairs have both the lines common. So $a = -3$ and the slope m of the line common to both the pairs is 1. Now,

$$\therefore x^2 + 2xy + ay^2 = x^2 + 2xy - 3y^2 = (x - y)(x + 3y)$$

and $ax^2 + 2xy + y^2 = -3x^2 + 2xy + y^2 = -(x - y)(3x + y)$

So, the equation of the required pair of lines is

$$(x + 3y)(3x + y) = 0$$

or $3x^2 + 10xy + 3y^2 = 0$

7. (1) Let the common line be $y = mx$. Then it must satisfy both the equations. Therefore, we have

$$bm^2 + 2hm + a = 0$$
 (1)

$$b'm^2 + 2h'm + a' = 0$$
 (2)

Exercises

Single Correct Answer Type

1. (2) Being a pair of lines, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. This gives $m = 4$. Now, find the angle between the lines.

Solving (1) and (2), we get

$$\frac{m^2}{2(ha' - h'a)} = \frac{m}{ab' - a'b} = \frac{1}{2(bh' - b'h)}$$

Eliminating m , we get

$$\left[\frac{ab' - a'b}{2(bh' - b'h)} \right]^2 = \frac{ha' - h'a}{bh' - b'h}$$

$$\text{or } (ab' - a'b)^2 = 4(ha' - h'a)(bh' - b'h)$$

8. (2) The given equation of the pair of straight lines can be rewritten as

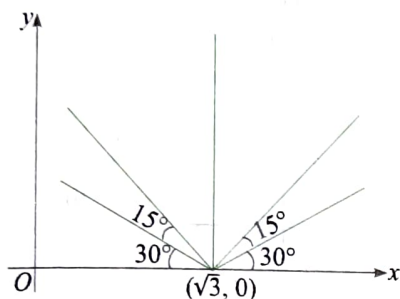
$$(\sqrt{3}y - x + \sqrt{3})(\sqrt{3}y + x - \sqrt{3}) = 0$$

Their separate equations are

$$\sqrt{3}y - x + \sqrt{3} = 0 \text{ and } \sqrt{3}y + x - \sqrt{3} = 0$$

$$\text{or } y = \frac{1}{\sqrt{3}}x - 1 \text{ and } y = -\frac{1}{\sqrt{3}}x + 1$$

$$\text{or } y = (\tan 30^\circ)x - 1 \text{ and } y = (\tan 150^\circ)x + 1$$



After rotation through an angle of 15° as given in the question, the lines are

$$(y - 0) = \tan 45^\circ (x - \sqrt{3}) \text{ and } (y - 0) = \tan 135^\circ (x - \sqrt{3})$$

$$\text{or } y = x - \sqrt{3} \text{ and } y = -x + \sqrt{3}$$

Their combined equation is

$$(y - x + \sqrt{3})(y + x - \sqrt{3}) = 0 \text{ or } y^2 - x^2 + 2\sqrt{3}x - 3 = 0$$

9. (3) We have $6x^2 - xy - 12y^2 = 0$

$$\text{or } (2x - 3y)(3x + 4y) = 0 \quad (1)$$

$$\text{and } 15x^2 + 14xy - 8y^2 = 0$$

$$\text{or } (5x - 2y)(3x + 4y) = 0 \quad (2)$$

Equation of the line common to (1) and (2) is

$$3x + 4y = 0 \quad (3)$$

Equation of any line parallel to (3) is

$$3x + 4y = k$$

Since its distance from (3) is 7, we have

$$\left| \frac{k}{\sqrt{3^2 + 4^2}} \right| = 7 \text{ or } k = \pm 35$$

10. (3) We have

$$x^2y^2 - 9y^2 - 6x^2y + 54y = 0$$

$$\text{or } y^2(x^2 - 9) - 6y(x^2 - 9) = 0$$

$$\text{or } y(y - 6)(x - 3)(x + 3) = 0$$

$$\text{or } y = 0, y = 6, x = 3, x = -3$$

So, the given equation represents four straight lines which form a square.

11. (3) The given equations are

$$a^2x^2 + 2h(a+b)xy + b^2y^2 = 0 \quad (1)$$

$$\text{and } ax^2 + 2hxy + by^2 = 0 \quad (2)$$

The equation of the bisectors of the angles between the lines represented by (1) is

$$\frac{x^2 - y^2}{a^2 - b^2} = \frac{xy}{h(a+b)}$$

$$\text{or } \frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

which is the same as the equation of bisectors of the angles between the line pair (2). Thus, two line pairs are equally inclined to each other.

12. (3) The equations of the given lines are

$$y - 1 = \tan \theta (x - 1)$$

$$\text{and } y - 1 = \cot \theta (x - 1)$$

So, their joint equation is

$$[(y - 1) - \tan \theta (x - 1)][(y - 1) - \cot \theta (x - 1)] = 0$$

$$\text{or } (y - 1)^2 - (\tan \theta + \cot \theta)(x - 1)(y - 1) + (x - 1)^2 = 0$$

$$\text{or } x^2 - (\tan \theta + \cot \theta)xy + y^2 + (\tan \theta + \cot \theta - 2)(x + y - 1) = 0$$

Comparing with the given equation, we get

$$\tan \theta + \cot \theta = a + 2$$

$$\text{or } \frac{1}{\sin \theta \cos \theta} = a + 2$$

$$\text{or } \sin 2\theta = \frac{2}{a + 2}$$

13. (4) Since the product of the slope of the four lines represented by the given equation is 1 and a pair of lines represents the bisectors of the angles between the other two, the product of the slopes of each pair is -1 .

So, let the equation of one pair be $ax^2 + 2hxy - ay^2 = 0$. Then the equation of its bisectors is

$$\frac{x^2 - y^2}{2a} = \frac{xy}{h}$$

By hypothesis,

$$x^4 + x^3y + cx^2y^2 - xy^3 + y^4 = (ax^2 + 2hxy - ay^2)(hx^2 - 2axy - hy^2)$$

$$= ah(x^4 + y^4) + 2(h^2 - a^2)(x^3y - xy^3) - 6ahx^2y^2$$

14. (2) Given $AB = BC$. Also,

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b}$$

$$\tan \theta = \frac{AB}{OA} = m_1$$

$$\text{and } m_2 = \tan \alpha = \frac{2AB}{OA} = 2m_1$$

$$\therefore \frac{m_2}{m_1} = 2$$

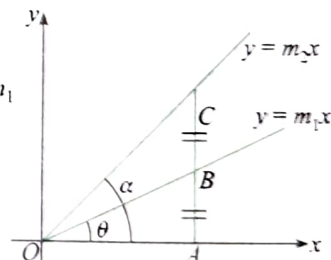
$$\text{or } \frac{m_2 + m_1}{m_2 - m_1} = \frac{2 + 1}{2 - 1} = 3$$

$$\text{or } -\frac{2h/b}{\sqrt{(4h^2/b^2) - (4a/b)}} = 3$$

$$\text{or } \frac{4h^2}{b^2} \cdot \frac{b^2}{4(h^2 - ab)} = 9$$

$$\text{or } 9h^2 - 9ab = h^2$$

$$\text{or } 8h^2 = 9ab$$



15. (4) Each point (x, y) has an image in line $y = 0$ as $(x, -y)$.

So, replacing y by $-y$ in the given equation, we get the image as $ax^2 - 2hxy + by^2 = 0$.

16. (2) The given pair of lines is

$$135x^2 - 136xy + 33y^2 = 0 \quad (1)$$

The equation of bisectors of angles between the pair of lines (1) is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \text{ or } \frac{x^2 - y^2}{135 - 33} = \frac{xy}{-68}$$

$$\text{or } 2x^2 + 3xy - 2y^2 = 0 \quad (2)$$

$$\text{or } (x + 2)(2x - y) = 0$$

One of the bisectors is $x + 2y = 0$ which is parallel to the line $x + 2y = 7$.

Hence, the line $x + 2y = 7$ is equally inclined to the given lines.

17. (2) Let m and m^2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$. Then,

$$m + m^2 = -\frac{2h}{b}$$

$$\text{and } mm^2 = \frac{a}{b} \text{ or } m^3 = \frac{a}{b}$$

$$\text{Now, } (m + m^2)^3 = \left(-\frac{2h}{b}\right)^3$$

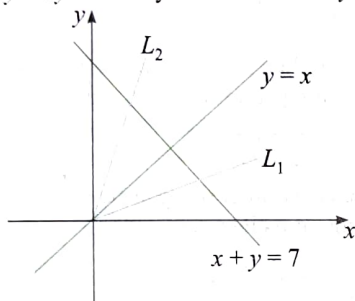
$$\therefore m^3 + m^6 + 3mm^2(m + m^2) = -\frac{8h^3}{b^3}$$

$$\therefore \frac{a}{b^2}(a+b) + \frac{8h^3}{b^3} = \frac{6ah}{b^2}$$

$$\therefore \frac{a+b}{h} + \frac{8h^2}{ab} = 6$$

These are the set of parallel lines and the distance between parallel lines are equal. So, the figure is a rhombus.

18. (1) $ax^2 + 2hxy + ay^2 = 0$ is symmetrical about $y = x$.



19. (2) Put $2h = -(a+b)$ in $ax^2 + 2hxy + by^2 = 0$. Then,

$$ax^2 - (a+b)xy + by^2 = 0$$

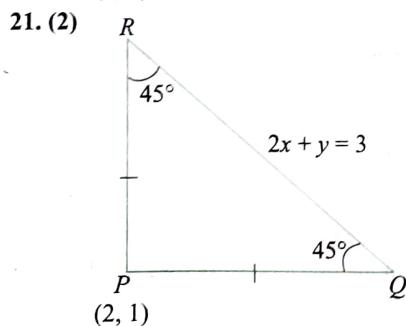
$$\text{or } (x-y)(ax-by) = 0$$

Therefore, one of the lines bisects the angle between the coordinates axes in the positive quadrant. Also, putting $b = -2h - a$ in $ax - by$, we have

$$ax - by = ax - (-2h - a)y = ax + (2h + a)y$$

Hence, $ax + (2h + a)y$ is a factor of $ax^2 + 2hxy + by^2$. However, statement 2 is not the correct explanation of statement 1.

20. (3) The lines by which triangle is formed are $x = 0$, $y = 0$, and $x + y = 1$. Clearly, it is a right triangle. We know that in a right-angled triangle, the orthocenter coincides with the vertex at which right angle is formed. Therefore, the orthocenter is $(0, 0)$.



Let m be the slope of PQ . Then,

$$\tan 45^\circ = \left| \frac{m - (-2)}{1 + m(-2)} \right| = \left| \frac{m+2}{1-2m} \right|$$

$$\therefore m+2 = 1-2m \text{ or } -1+2m = m+2$$

$$\therefore m = -\frac{1}{3} \text{ or } m = 3$$

Hence, the equation of PQ is

$$y - 1 = -\frac{1}{3}(x - 2)$$

$$\text{or } x + 3y - 5 = 0$$

and the equation of PR is

$$y - 1 = 3(x - 2)$$

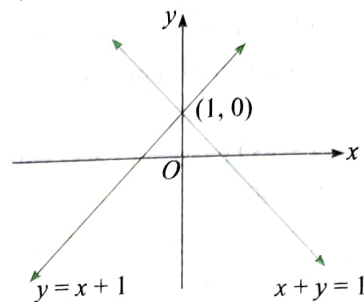
$$\text{or } 3x - y - 5 = 0$$

Hence, the combined equation of PQ and PR is

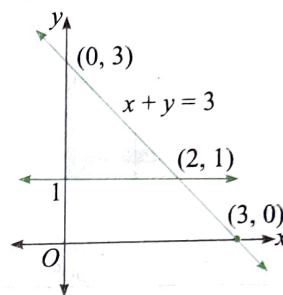
$$(x + 3y - 5)(3x - y - 5) = 0$$

$$\text{or } 3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$$

22. (1) $x^2 - y^2 + 2y = 1$ or $x = \pm(y - 1)$



The bisectors of the above lines are $x = 0$ and $y = 1$.



So, the area between $x = 0$, $y = 1$, and $x + y = 3$ is the shaded region shown in the figure. The area is given by $(1/2) \times 2 \times 2 = 2$ sq. units.

23. (1) The component lines of given pair of straight lines are perpendicular to each other. So, orthocentre is the point of intersection of component lines.

Multiple Correct Answers Type

1. (1), (2), (3)

The equation is

$$x^2(x+y) - y^2(x+y) = 0$$

$$\text{or } (x+y)^2(x-y) = 0$$

It represents the lines $x + y = 0$, $x + y = 0$, $x - y = 0$.

2. (1), (3)

The equation represents a pair of straight lines. Hence,

$$1 \times (-2)(-1) + 2\left(\frac{3}{2}\right) \times 0 \times \frac{m}{2} - 1 \times \left(\frac{3}{2}\right)^2 - (-2)$$

$$\times 0^2 - (-1) \times \left(\frac{m}{2}\right)^2 = 0$$

$$\text{or } m = 1, -1$$

The points of intersection of the pair of lines are obtained by solving

$$\frac{\partial S}{\partial x} \equiv 2x + my = 0$$

$$\text{and } \frac{\partial S}{\partial y} \equiv mx - 4y + 3 = 0$$

When $m = 1$, the required point is the intersection of $2x + y = 0$, $x - 4y + 3 = 0$. When $m = -1$, the required point is the intersection of $2x - y = 0$, $-x - 4y + 3 = 0$.

3. (1), (4)

$$\text{Here, } my(y - mx) + x(y - mx) = 0$$

$$\text{or } (y - mx)(my + x) = 0$$

So, the lines are $y = mx$ or $y = (-1/m)x$. The bisectors between the lines $xy = 0$ are $y = x$ and $y = -x$. Therefore, $m = 1, -1$.

4. (1), (2), (4)

Let the equation of the lines given by $x^2 + 2hxy + y^2 = 0$ be $y = m_1x$ and $y = m_2x$. Since these make an angle α with $y + x = 0$ whose slope is -1 , we have

$$\frac{m_1 + 1}{1 - m_1} = \tan \alpha = \frac{-1 - m_2}{1 - m_2}$$

$$\text{or } m_1 + m_2 = \frac{(\tan \alpha - 1)^2 + (\tan \alpha + 1)^2}{\tan^2 \alpha - 1}$$

$$= \frac{-2 \sec^2 \alpha \times \cos^2 \alpha}{\cos 2\alpha}$$

$$\therefore -2 \sec 2\alpha = -2h$$

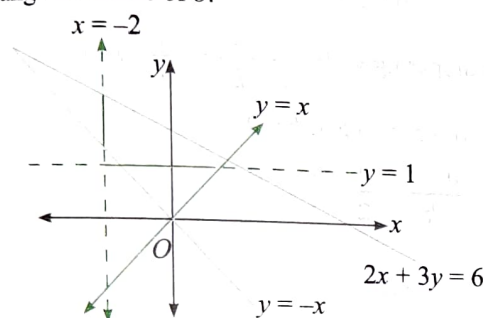
$$\text{or } \sec 2\alpha = h$$

$$\text{or } \cos 2\alpha = \frac{1}{h} \text{ or } 2 \cos^2 \alpha - 1 = \frac{1}{h}$$

$$\text{or } \cos \alpha = \sqrt{\frac{1+h}{2h}} \text{ and } \cot \alpha = \sqrt{\frac{h+1}{h-1}}$$

5. (1), (4)

The separate equations of the sides are $x - y = 0$, $x + y = 0$, and $2x + 3y - 6 = 0$. The point $(-2, a)$ moves on the line $x = -2$ and the point $(b, 1)$ moves on the line $y = 1$. From the figure, the y -coordinates of the points of intersection of $x = -2$ with $y = -x$ and $2x + 3y = 6$ give the range of values of a . The x -coordinates of the points of intersection of $y = 1$ with $y = -x$ and $y = x$ give the range of values of b .



6. (1), (2)

$$\text{Let } \frac{2}{3}x^2 + \frac{p}{3}xy + y^2 = (y - mx)(y - m'x)$$

$$\text{and } \frac{2}{-3}x^2 + \frac{q}{-3}xy + y^2 = (y + \frac{1}{m}x)(y - m'x)$$

Then,

$$m + m' = -\frac{p}{3}, mm' = \frac{2}{3} \quad (1)$$

$$\frac{1}{m} - m' = -\frac{q}{3}, -\frac{m'}{m} = -\frac{2}{3} \quad (2)$$

$$\text{or } m^2 = 1 \text{ or } m = \pm 1$$

If $m = 1$, $m' = 2/3$ and, so, $p = -5$, $q = -1$. If $m = -1$, $m' = -2/3$ and, so, $p = 5$, $q = 1$.

7. (1), (2), (3), (4)

The equation of the lines joining the origin to the points of intersection of the given lines is

$$3x^2 + mxy - 4x(2x + y) + 1(2x + y)^2 = 0 \quad (\text{by homogenization})$$

$$\text{or } x^2 - mxy - y^2 = 0$$

which are perpendiculars for all values of m .

8. (2), (4)

m and m^2 are the roots of equation

$$\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + a = 0$$

$$\therefore m + m^2 = 6 \text{ and } m^3 = a$$

$$\text{or } m^3 + m^6 + 3m^3(m + m^2) = 216$$

$$a + a^2 + 3a \cdot 6 = 216$$

$$a^2 + 19a - 216 = 0 \text{ or } a = 8, a = -27$$

9. (2), (4)

The equation $y^2 + xy - 12x^2 = 0$ can be rewritten as

$$(y + 4x)(y - 3x) = 0$$

$$\text{or } \frac{y}{x} = -4, 3$$

The two pairs will have a line common if -4 or 3 is a root of

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

$$\therefore 9b + 6h + a = 0 \text{ or } 16b - 8h + a = 0$$

Linked Comprehension Type

$$1. (2) \quad a = \lambda; h = -5; b = 12; g = \frac{5}{2}; f = -8, c = -3$$

$$\therefore \lambda(12)(-3) + 2(-8)\left(\frac{5}{2}\right)(-5) - \lambda(64) - \left(\frac{25}{4}\right) \times 12 + 3 \times 25 = 0$$

$$\text{or } -36\lambda + 200 - 64\lambda - 75 + 75 = 0$$

$$\text{or } 100\lambda = 200$$

$$\text{or } \lambda = 2$$

$$2. (1) \quad 2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$$

Consider the homogeneous part

$$2x^2 - 10xy + 12y^2 = (x - 2y)(2x - 6y)$$

$$\therefore 2x^2 - 10xy + 12y^2 + 5x - 16y - 3 =$$

$$\equiv (2x - 6y + A)(x - 2y + B)$$

Comparing coefficients, we get

$$A = -1, B = 3$$

Hence, the lines are

$$2x - 6y - 1 = 0 \quad \text{and} \quad x - 2y + 3 = 0$$

Solving, we get the intersection point as $(-10, -7/2)$. Therefore, Product = 35

$$3. (3) \quad \tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{25 - 24}}{14} = \frac{1}{7}$$

4. (3), 5. (1), 6. (4)

As the lines are perpendicular,

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\therefore a - 2 = 0$$

$$\text{or } a = 2$$

$$\text{Also, } abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

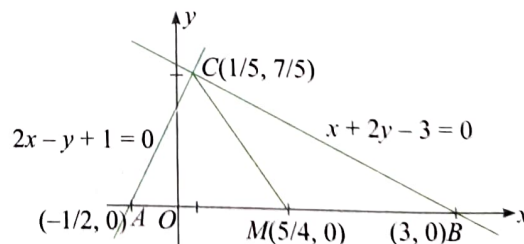
$$\therefore c = -3$$

Hence, the given pair of lines is

$$2x^2 + 3xy - 2y^2 - 5x + 5y - 3 = 0$$

Factorizing, we get lines

$$x + 2y - 3 = 0 \text{ and } 2x - y + 1 = 0$$



The point of intersection of the lines is $C(1/5, 7/5)$.

The points of intersection of the lines with the x -axis are $A(3, 0)$ and $B(-1/2, 0)$.

The orthocenter of triangle is $C(1/5, 7/5)$ and the circumcenter is the midpoint of AB which is $M(5/4, 0)$. Therefore,

$$CM = \sqrt{\left(\frac{5}{4} - \frac{1}{5}\right)^2 + \frac{49}{25}} = \frac{7}{4}$$

Numerical Value Type

1. (2) $(x + 7y)^2 + 4\sqrt{2}(x + 7y) - 42 = 0$

or $(x + 7y)^2 + 7\sqrt{2}(x + 7y) - 3\sqrt{2}(x + 7y) - 42 = 0$

or $(x + 7y)[x + 7y + 7\sqrt{2}] - 3\sqrt{2}(x + 7y + 7\sqrt{2}) = 0$

or $(x + 7y + 7\sqrt{2})(x + 7y - 3\sqrt{2}) = 0$

or $x + 7y + 7\sqrt{2} = 0$ and $x + 7y - 3\sqrt{2} = 0$

or $d = \left| \frac{7\sqrt{2} + 3\sqrt{2}}{\sqrt{1 + 49}} \right| = \frac{10\sqrt{2}}{\sqrt{50}} = 2$

2. (3) We have

$$y^2 - 9xy + 18x^2 = 0$$

or $y^2 - 6xy - 3xy + 18x^2 = 0$

or $y(y - 6x) - 3x(y - 6x) = 0$

or $y - 3x = 0$ and $y - 6x = 0$

The third line is $y = 6$. Therefore, the area of the triangle formed by these lines

$$= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 6 & 1 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= \frac{1}{2} |6 - 12|$$

$$= 3 \text{ units}^2$$

3. (2) The equation $4x^2 + 8xy + ky^2 - 9 = 0$ represents a pair of straight lines if $(4)(k)(-9) - (-9)(4)^2 = 0$ or $k = 4$.

4. (0) Let m_1 and m_2 be the slopes of the lines $x^2 + hxy + 2y^2 = 0$. Then,

$$m_1 + m_2 = -\frac{h}{2}, m_1 m_2 = \frac{1}{2}$$

But $m_1 = 2m_2$ (given). Therefore,

$$3m_2 = -\frac{h}{2} \text{ and } 2m_2^2 = \frac{1}{2}$$

i.e., $m_2^2 = \frac{1}{4}$

Also, $m_2 = -\frac{h}{6}$

$$\therefore \frac{h^2}{36} = \frac{1}{4} \text{ or } h^2 = 9 \text{ or } h = \pm 3$$

5. (1.5)

Shifting the origin to $(1, 2)$, the equation of given pair of lines becomes

$$ax^2 + 2hxy + by^2 = 0$$

whose one angle bisector is $x + 2y = 0$ (given) and other is $2x - y = 0$.

$\therefore$ Combined equation of angle bisector is

$$(x + 2y)(2x - y) = 0$$

On comparing with $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, we get

$$a - b = -3 \text{ and } h = 2$$

$$\therefore \frac{b - a}{h} = \frac{3}{2}$$

Chapter 4

Concept Application Exercises

Exercise 4.1

1. Radius = Perpendicular distance from $(1, -3)$ to $(3x - 4y - 5 = 0)$

$$= \left| \frac{3 + 12 - 5}{\sqrt{5^2}} \right| = 2.$$

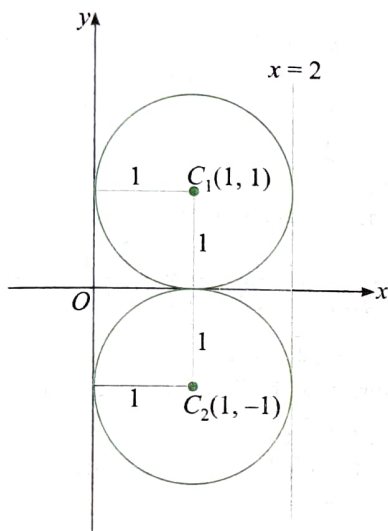
2. The center is $(1, 2)$ and since the circle touches the x -axis, the radius is equal to 2.

Hence, the equation is

$$(x - 1)^2 + (y - 2)^2 = 2^2$$

or $x^2 + y^2 - 2x - 4y + 1 = 0$

3.



Clearly, radius of the circle is '1' and centre can be $(1, \pm 1)$.

Therefore, equation of circle will be $(x - 1)^2 + (y \pm 1)^2 = 1$.

4. Lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$ are two parallel tangents.

Diameter is distance between these two lines, which is

$$\frac{|10 - (-30)|}{\sqrt{(4)^2 + (-3)^2}} = 8 \text{ units.}$$

$\therefore$ Radius = 4 units

Centre lies on the line $2x + y = 0$, so let the centre be $C(h, -2h)$.

Now, distance of C from any of tangent lines is 4.

$$\therefore \frac{|4h + 6h + 10|}{\sqrt{(4)^2 + (-3)^2}} = 4$$

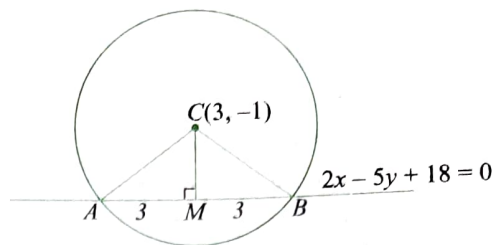
$$\therefore |h + 1| = 2$$

$$\therefore h = 1$$

Therefore, centre is $C(1, -2)$ (as centre lies between the given lines).

5. Let the circle be as shown below.

$$CM = \frac{|2(3) - 5(-1) + 18|}{\sqrt{2^2 + (-5)^2}} = \sqrt{29}$$

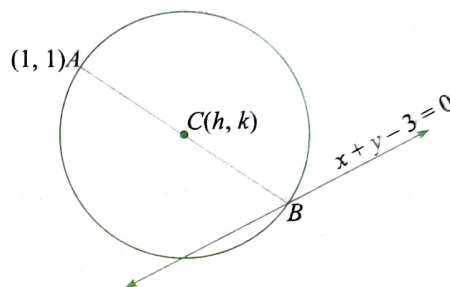


$$\therefore AC^2 = AM^2 + CM^2 = 9 + 29 = 38$$

$$\therefore \text{Radius} = \sqrt{38}$$

Hence, equation of circle is $(x - 3)^2 + (y + 1)^2 = 38$.

6. Let $C(h, k)$ be the centre and AB be the diameter of the circle.



$$\therefore B \equiv (2h - 1, 2k - 1)$$

Point B lies on the line $x + y - 3 = 0$.

$$\therefore 2h - 1 + 2k - 1 - 3 = 0$$

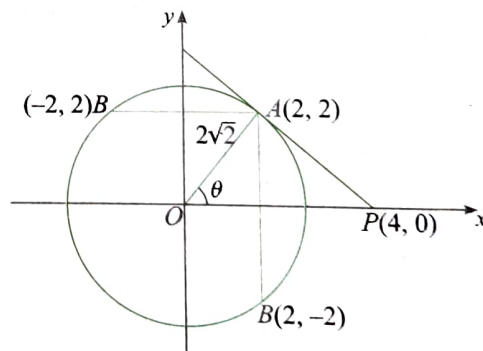
Hence, required locus is $2x + 2y - 5 = 0$.

$$7. \cos \theta = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

$$\therefore A \equiv (2, 2)$$

Let $B \equiv (x_1, y_1)$.



Given $AB = 4$. Therefore,

$$(x_1 - 2)^2 + (y_1 - 2)^2 = 16$$

$$\text{or } x_1^2 + y_1^2 - 4x_1 - 4y_1 = 8$$

$$\text{Also, } x_1^2 + y_1^2 = 8$$

$$\therefore x_1 + y_1 = 0$$

$$\therefore 2x_1^2 = 8$$

$$\text{or } x_1 = \pm 2$$

$$\therefore B \equiv (2, -2) \text{ or } (-2, 2)$$

8. Here the center of the circle, $(3, -1)$, must lie on the line $x + 2by + 7 = 0$. Therefore,

$$3 - 2b + 7 = 0 \text{ or } b = 5$$

9. Here, $g = 5$ and $c = 9$.

$$\therefore \text{Length of intercept} = 2\sqrt{g^2 - c} = 2\sqrt{25 - 9} = 8$$

10. The center of the given circle is $(1, 2)$. Let (α, β) be the other end. Then,

$$\frac{\alpha + 3}{2} = 1; \frac{\beta + 2}{2} = 2$$

$$\text{or } \alpha = -1, \beta = 2$$

Therefore, the other end is $(-1, 2)$.

11. Let vertices of triangle are $(0, 0)$, $(a, 0)$ and (b, c) .

The variable point (x, y) is such that

$$(x - 0)^2 + (y - 0)^2 + (x - a)^2 + (y - 0)^2 + (x - b)^2 + (y - c)^2 = \lambda \text{ (constant)}$$

$$\therefore 3x^2 + 3y^2 - 2(a + b)x - 2cy + a^2 + b^2 + c^2 - \lambda = 0, \text{ which is equation of circle.}$$

12. Radius ≤ 5

$$\sqrt{\left(\frac{\lambda}{2}\right)^2 + \left(\frac{1 - \lambda}{2}\right)^2} - 5 \leq 5$$

$$\Rightarrow 2\lambda^2 - 2\lambda - 119 \leq 0$$

$$\Rightarrow \frac{1 - \sqrt{239}}{2} \leq \lambda \leq \frac{1 + \sqrt{239}}{2}$$

$$\Rightarrow -7.2 \leq \lambda \leq 8.2 \text{ (approximately)}$$

$$\Rightarrow \lambda = -7, -6, -5, \dots, 7, 8$$

Also, we must have

$$\left(\frac{\lambda}{2}\right)^2 + \left(\frac{1 - \lambda}{2}\right)^2 - 5 \geq 0$$

$$\Rightarrow 2\lambda^2 - 2\lambda - 19 \geq 0$$

$$\Rightarrow \lambda \leq \frac{1 - \sqrt{39}}{2}$$

$$\text{or } \lambda \leq \frac{1 + \sqrt{39}}{2}$$

From (1) and (2),

$$\lambda = -7, -6, -5, -4, -3, 4, 5, 6, 7, 8$$

13. Let (h, k) be the centroid. Then,

$$h = \frac{a \cos t + b \sin t + 1}{3}$$

$$\text{and } k = \frac{a \sin t - b \cos t + 0}{3}$$

$$\text{or } 3h - 1 = a \cos t + b \sin t$$

$$\text{and } 3k = a \sin t - b \cos t$$

Squaring and adding (1) and (2), we get

$$(3h - 1)^2 + (3k)^2 = a^2 + b^2$$

Hence, the locus of (h, k) is $(3x - 1)^2 + (3y)^2 = a^2 + b^2$.

14. Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Therefore, the intercept on the x -axis is $2\sqrt{g^2 - c}$.

Also, the intercept on the y -axis is $2\sqrt{f^2 - c}$.

According to the question,

$$\sqrt{g^2 - c} = 2\sqrt{f^2 - c}$$

$$\text{or } g^2 - 4f^2 = -3c$$

Given, radius = 2

$$\therefore g^2 + f^2 - c = 4$$

Eliminating c from (1) and (2), we get

$$4g^2 - f^2 = 12$$

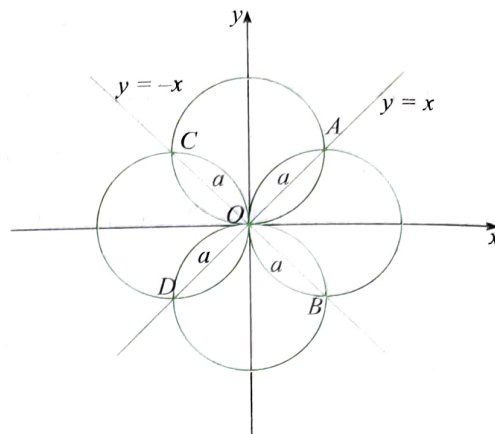
Therefore, the locus is $4x^2 - y^2 = 12$.

Exercise 4.2

1. The extremities of diameter are $(5, 7)$ and $(1, 4)$. The radius is half of the distance between them, i.e.,

$$\frac{1}{2} \sqrt{(5 - 1)^2 + (7 - 4)^2} = \frac{5}{2}$$

2.



In the figure, length of chords OA and AB that cut off by the circles is a .

$\therefore A \equiv \left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and $B \equiv \left(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}\right)$; which are end points of diameters.

Therefore, the equation of circle is

$$\left(x - \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0$$

$$\text{or } x^2 + y^2 - \sqrt{2}ax = 0$$

Similarly, circle with C and D as end points of diameter is $x^2 + y^2 + \sqrt{2}ax = 0$.

Circles with AC and BD as diameter are given by $x^2 + y^2 \pm \sqrt{2}ay = 0$.

3. Obviously, $(3, 0)$ and $(0, 4)$ are the endpoints of diameter.

Then, the equation is $(x - 3)(x - 0) + (y - 0)(y - 4) = 0$, i.e.,

$$x^2 + y^2 - 3x - 4y = 0.$$

4. The circle passing through $(1, 0)$, $(0, 1)$, and $(0, 0)$ is

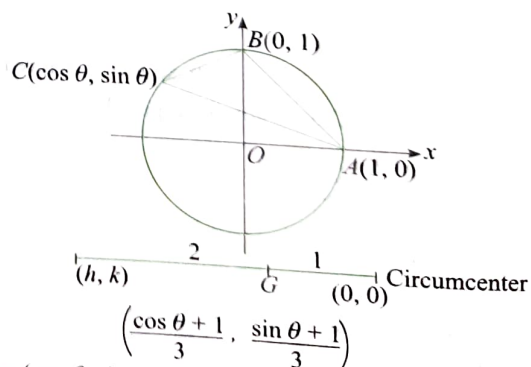
$$x(x - 1) + y(y - 1) = 0$$

It also passes through the point $(2k, 3k)$. Therefore,

$$2k(2k - 1) + 3k(3k - 1) = 0$$

$$\text{i.e., } k = 0 \text{ or } k = \frac{5}{13}$$

5.



Let $C \equiv (\cos \theta, \sin \theta)$; $H(h, k)$ is the orthocenter of $\triangle ABC$.

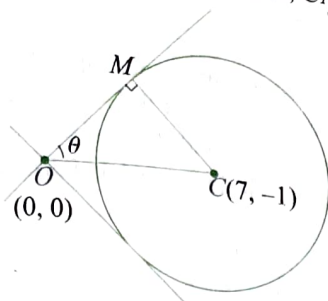
Since the circumcenter of the triangle is $(0, 0)$, for the orthocenter,

Eliminating θ ,

$$(x - 1)^2 + (y - 1)^2 = 1$$

$$\text{or } x^2 + y^2 - 2x - 2y + 1 = 0$$

1. Centre of the circle is $C(7, -1)$ and radius, $CM = 5$.



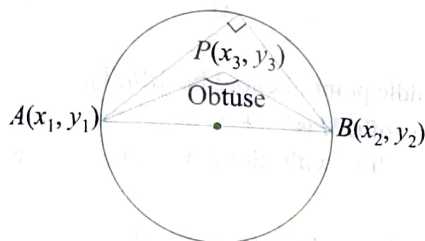
$$OC = \sqrt{7^2 + (-1)^2} = 5\sqrt{2}$$

$$\therefore \sin \theta = \frac{CM}{OC} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = 45^\circ$$

Hence, angle between tangents is 90° .

2. The equation of the circle with AB as diameter is
 $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

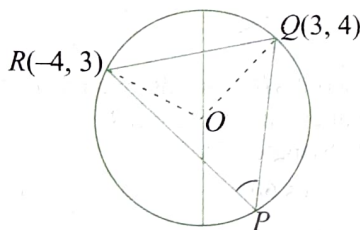


Since AB subtends an obtuse angle at (x_3, y_3) , P lies inside the circle. Therefore,

$$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) < 0$$

3. We know that

$$\angle QPR = \frac{1}{2} \angle QOR$$



O being the center $(0, 0)$ of the given circle $x^2 + y^2 = 25$.

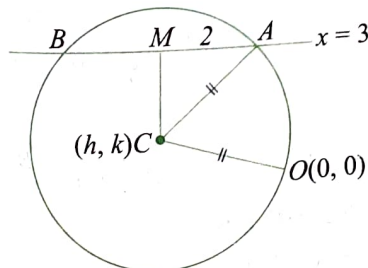
$$\text{Let } m_1 = \text{Slope of } OQ = \frac{4}{3}$$

$$\text{and } m_2 = \text{Slope of } OR = -\frac{3}{4}$$

As $m_1 m_2 = -1$, we have

$$\angle QOR = \frac{\pi}{2} \text{ or } \angle QPR = \frac{\pi}{4}$$

- 4.



From the figure,

$$CO = CA$$

$$\text{or } CO^2 = CM^2 + AM^2$$

$$\Rightarrow (h - 0)^2 + (k - 0)^2 = \left(\frac{|h - 3|}{1}\right)^2 + 4$$

$$\Rightarrow h^2 + k^2 = h^2 - 6h + 13$$

$$\text{or } k^2 = -6h + 13$$

$$\text{or } k^2 + 6h = 13$$

Hence, the required locus is $y^2 + 6x = 13$.

$$5. 16x^2 + 16y^2 + 48x - 8y - 43 = 0$$

$$\text{or } x^2 + y^2 + 3x - \frac{1}{2}y - \frac{43}{16} = 0$$

$$\text{Center of the circle : } \left(-\frac{3}{2}, \frac{1}{4}\right)$$

$$\text{Radius} = \sqrt{\frac{9}{4} + \frac{1}{16} + \frac{43}{16}} = \sqrt{5}$$

Required least distance

= Distance of line from centre of circle - Radius of circle

$$= \left| 8\left(-\frac{3}{2}\right) - 4\left(\frac{1}{4}\right) + 73 \right|$$

$$= \frac{60}{4\sqrt{5}} - \sqrt{5} = 2\sqrt{5}$$

6. According to the question,

$$\sqrt{(5)^2 + (3)^2 + 2(5) + k(3) + 17} = 7$$

$$\text{or } 61 + 3k = 49 \text{ or } k = -4$$

7. From the figure,

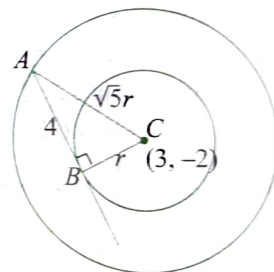
$$5r^2 = 16 + r^2$$

$$\therefore r^2 = 4$$

$$\therefore r = 2$$

$\therefore$ Area between circles

$$= \pi(5r^2) - \pi(r^2) = 4\pi r^2 = 16\pi$$



8. Let the variable point be (h, k) .

According to the question,

$$\frac{\sqrt{h^2 + k^2 + 4h + 3}}{\sqrt{h^2 + k^2 - 6h + 5}} = \frac{2}{3}$$

Squaring and simplifying, we get

$$5h^2 + 5k^2 + 60h + 7 = 0$$

Hence, the required locus is $5x^2 + 5y^2 + 60x + 7 = 0$.

9. Let (x_1, y_1) be any point on the circle

$$x^2 + y^2 + 2gx + 2fy + c_1 = 0$$

$$\therefore x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_1 = 0$$

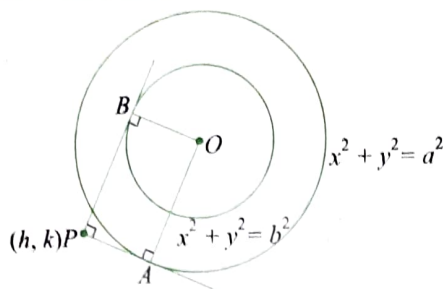
The length of the tangent from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c_2 = 0$ is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c_2} = \sqrt{c_2 - c_1} \quad [\text{Using (1)}]$$

10. In the figure, from variable point $P(h, k)$ tangents PA and PB are drawn to two circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$. Tangents PA and PB are perpendicular.

Clearly, $OAPB$ is a rectangle.

$$\therefore OP^2 = OB^2 + BP^2 = h^2 + k^2 = a^2 + b^2$$



Therefore, required locus is $x^2 + y^2 = a^2 + b^2$, which is concentric circle with given circles.

11. Chord through $P(1, 1)$ meets the circle at A and B .

$$\therefore PA \times PB = (\text{length of tangent})^2 = PT^2$$

$$\text{or } PA \times PB = 2 - r^2 \quad (1)$$

$$\text{Given } \frac{PB}{PA} = \frac{\sqrt{2} + r}{\sqrt{2} - r} \quad (2)$$

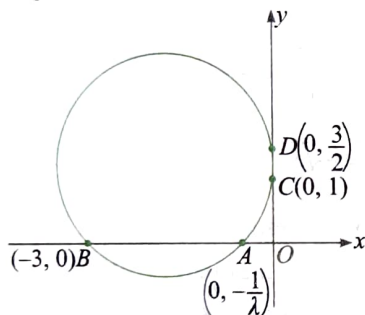
Solving (1) and (2), we get

$$PB = \sqrt{2} + r, PA = \sqrt{2} - r$$

$$\Rightarrow AB = PB - PA = 2r = \text{diameter of circle}$$

Therefore, equation of chord AB is $y = x$.

12.



Line $\lambda x - y + 1 = 0$ meets x -axis at $A(-1/\lambda, 0)$ and y -axis at $C(0, 1)$.

Line $x - 2y + 3 = 0$ meets x -axis at $B(-3, 0)$ and y -axis at $D(0, 3/2)$.

Since points B, A, C and D lie on a circle, we must have

$$OA \times OB = OC \times OD$$

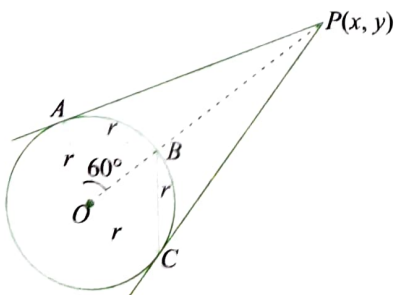
$$\Rightarrow \frac{1}{\lambda} \times 3 = 1 \times \frac{3}{2}$$

$$\Rightarrow \lambda = 2$$

13. Let the point of intersection of tangents at A and C be $P \equiv (x, y)$.

Since $AB = AO = BO = r$, we get

$$\angle AOB = 60^\circ$$



$$\text{or } \frac{PA}{r} = \tan 60^\circ = \sqrt{3}$$

$$\text{or } \sqrt{x^2 + y^2 - r^2} = \sqrt{3}r$$

Therefore, $x^2 + y^2 = 4r^2$ is the locus of the point P .

14. We must have

Radius of given circle > Perpendicular distance from the center of circle to the given line

$$\text{or } \sqrt{4 + 16 + 5} > \frac{|3(2) - 4(4) - m|}{\sqrt{9 + 16}}$$

$$\text{or } |m + 10| < 25$$

$$\text{or } -35 < m < 15$$

15. If d is the common difference of AP , then the radius of the smallest circle is $1 - 2d$. If the given line $y - x - 1 = 0$ cuts the smallest circle at real and distinct points, then it will definitely cut the remaining circles at real and distinct points. Therefore,

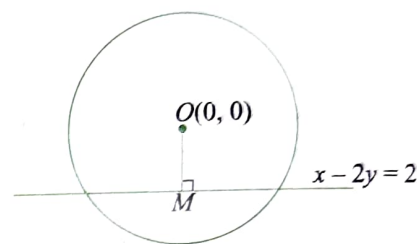
$$\frac{|0 - 0 - 1|}{\sqrt{2}} < (1 - 2d)$$

$$\text{or } 1 - 2d > \frac{1}{\sqrt{2}}$$

$$\text{or } d < \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right)$$

$$\text{Hence, } d \in \left(0, \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}} \right) \right)$$

- 16.

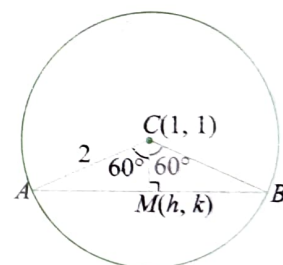


Let the middle point of chord be $M(h, k)$.

So, equation of OM is $2x + y = 0$.

Solving this line with chord $x - 2y = 2$, we get point M as $(2/5, -4/5)$.

17. Given circle is $(x - 1)^2 + (y - 1)^2 = 4$.



In the figure, chord AB subtends an angle of 120° at the centre.

$M(h, k)$ is the midpoint of chord AB .

In right angled triangle AMC ,

$$CM = AC \cos 60^\circ$$

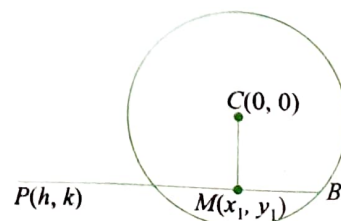
$$\therefore \sqrt{(h - 1)^2 + (k - 1)^2} = 2 \times \frac{1}{2}$$

Therefore, required locus is $(x - 1)^2 + (y - 1)^2 = 1$.

18. From the diagram,

$$CM \perp BP$$

$$\therefore \left(\frac{y_1}{x_1} \right) \left(\frac{y_1 - k}{x_1 - h} \right) = -1$$



So, the locus of M is

$$x^2 + y^2 = hx + ky$$

Exercise 4.4

1. Let the equation of the tangent be

$$\frac{x}{a} + \frac{y}{a} = 1$$

$$\text{i.e., } x + y = a$$

Therefore, the length of perpendicular from the center $(-2, 2)$ on (1) is equal to the radius, i.e., $\sqrt{4 + 4 - 4}$. So,

$$\frac{|-2 + 2 - a|}{\sqrt{1 + 1}} = 2$$

$$\text{or } a = \pm 2\sqrt{2}$$

Hence, the equations of the tangents are $x + y = \pm 2\sqrt{2}$.

2. Given circle is
- $x^2 + y^2 - 22x - 4y + 25 = 0$
- .

Centre is $(11, 2)$ and radius is $\sqrt{121 + 4 - 25} = 10$

Equation of line perpendicular to $5x + 12y + 8 = 0$ is $12x - 5y + k = 0$.

If this line touches the circle, then distance of the centre of the circle from the line is radius.

$$\therefore \left| \frac{12(11) - 5(2) + k}{\sqrt{144 + 25}} \right| = 10$$

$$\Rightarrow |k + 122| = 130$$

$$\Rightarrow k = 8 \text{ or } -252$$

Hence equations of tangents are

$$12x - 5y + 8 = 0 \text{ and } 12x - 5y = 252.$$

3. Distance of centre of the circle from the line must be radius of the circle.

$$\therefore a = \frac{|0 + 0 + n|}{\sqrt{l^2 + m^2}}$$

$$\therefore a^2(l^2 + m^2) = n^2$$

4. We have
- $S = x^2 + y^2 + 20x + 20y + 20 = 0$

Equation of pair of tangents from origin is given by $T^2 = SS_1$.

$$\text{or } (0 \cdot x + 0 \cdot y + 10(x + 0) + 10(y + 0) + 20)^2 = (x^2 + y^2 + 20x + 20y + 20)(20)$$

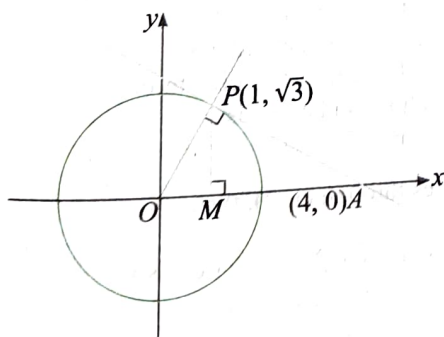
$$\text{or } 5(x + y + 2)^2 = (x^2 + y^2 + 20x + 20y + 20)$$

$$\text{or } 2x^2 + 2y^2 + 5xy = 0$$

5. Equation of tangent to circle
- $x^2 + y^2 = 4$
- at point
- $P(1, \sqrt{3})$
- is

$$1 \cdot x + \sqrt{3} \cdot y = 4, \text{ i.e. } x + \sqrt{3}y = 4.$$

It meets x -axis at $A(4, 0)$.



$$\therefore \text{Area of triangle } OAP = \frac{1}{2} OA \times PM$$

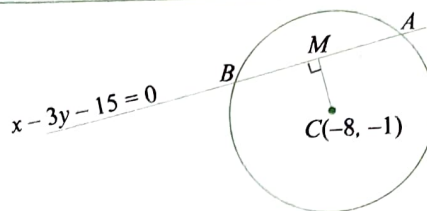
$$= \frac{1}{2} 4 \times \sqrt{3} = 2\sqrt{3}$$

6. Equation of tangent to circle
- $x^2 + y^2 - 4x + 2y - 5 = 0$
- at point
- $(3, -4)$
- is

$$3x - 4y - 2(x + 3) + (y - 4) - 5 = 0$$

$$\text{or } x - 3y - 15 = 0$$

(1)



Center of the second circle is $C(-8, -1)$

Slope of CM is -3

Equation of CM is $y + 1 = -3(x + 8)$

$$\text{or } 3x + y + 25 = 0$$

(2)

Solving (1) and (2), we get $M \equiv (-6, -7)$

7. The angle between
- $3x + y = 0$
- and the line joining
- $(2, -1)$
- to
- $(0, 0)$
- is

$$\theta = \tan^{-1} \left| \frac{-3 + \frac{1}{2}}{1 + (-3)\left(-\frac{1}{2}\right)} \right|$$

$$= \tan^{-1} |-1| = \frac{\pi}{4}$$

The other tangent is perpendicular to $3x + y = 0$.

Therefore, the equation of the other tangent is $x - 3y = 0$.

8. The equation of the circle is

$$x^2 + y^2 - 2x - 4y - 20 = 0$$

Center $\equiv (1, 2)$ and radius $= \sqrt{1 + 4 + 20} = 5$

The equation of tangent at $(1, 7)$ is

$$x \cdot 1 + y \cdot 7 - (x + 1) - 2(y + 7) - 20 = 0$$

$$\text{or } y - 7 = 0$$

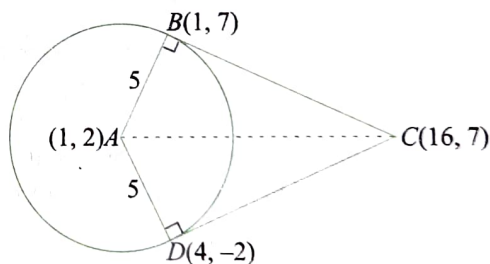
(1)

Similarly, the equation of tangent at $(4, -2)$ is

$$4x - 2y - (x + 4) - 2(y - 2) - 20 = 0$$

$$\text{or } 3x - 4y - 20 = 0$$

(2)



For point C , solving (1) and (2), we get $x = 16$ and $y = 7$. Therefore,

$$C \equiv (16, 7)$$

Now, clearly,

$$\begin{aligned} \text{ar}(\text{quad. } BCDA) &= 2 \times \text{ar}(\triangle ABC) \\ &= 2 \times \frac{1}{2} \times AB \times BC \\ &= AB \times BC \end{aligned}$$

where $AB = \text{Radius of circle} = 5$

and $BC = 15$

$$\therefore \text{ar}(\text{quad. } ABCD) = 5 \times 15 = 75 \text{ sq. units}$$

9. Clearly, the point
- $(1, 2)$
- is the center of the given circle and infinite tangents can only be drawn on a point circle.

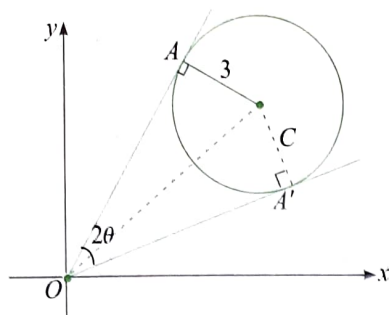
Hence, the radius should be 0. Therefore,

$$\sqrt{1^2 + 2^2} - \lambda = 0 \text{ or } \lambda = 5$$

10. The equation
- $2x^2 - 3xy + y^2 = 0$
- represents a pair of tangents
- OA
- and
- OA'
- .

Let the angle between these two tangents be 2θ . Then,

$$\tan 2\theta = \frac{2\sqrt{(-3/2)^2 - 2 \times 1}}{2 + 1}$$



$$\text{or } \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{1}{3}$$

$$\text{or } \tan^2 \theta + 6 \tan \theta - 1 = 0$$

$$\text{or } \tan \theta = \frac{-6 \pm \sqrt{36 + 4}}{2} = -3 \pm \sqrt{10}$$

As θ is acute, $\tan \theta = \sqrt{10} - 3$.

Now, we know that the line joining the point through which tangents are drawn to the center bisects the angle between the tangents. Therefore,

$$\angle AOC = \theta$$

In $\triangle OAC$,

$$\tan \theta = \frac{3}{OA}$$

$$\text{or } OA = \frac{3}{\sqrt{10} - 3} \times \frac{\sqrt{10} + 3}{\sqrt{10} + 3}$$

$$\therefore OA = 3(3 + \sqrt{10})$$

11. Clearly, $x^2 + y^2 = 2a^2$ is director circle of the circle $x^2 + y^2 = a^2$.

Hence, in the diagram, $ABOC$ is a square and circumcenter $P(h, k)$ of $\triangle ABC$ is the midpoint of OA . Hence,

$$\sqrt{h^2 + k^2} = \frac{\sqrt{2}a}{2}$$

or the locus is

$$x^2 + y^2 = \frac{a^2}{2}$$

12. Chord of contact from the origin $\equiv gx + fy + c = 0$
and that from $(g, f) \equiv gx + fy + g(x + g) + f(y + f) + c = 0$
or $2gx + 2fy + g^2 + f^2 + c = 0$

$$\therefore \text{Required distance} = \frac{\left| \frac{g^2 + f^2 + c}{2} - c \right|}{\sqrt{g^2 + f^2}} = \frac{|g^2 + f^2 - c|}{2\sqrt{g^2 + f^2}}$$

13. Let point P be (h, k) .

Equation of chord of contact of the given circle w.r.t. point P is

$$4hx + 4ky - 3(x + h) + 5(y + k) - 14 = 0$$

$$\text{or } (4h - 3)x + (4k + 5)y - 3h + 5k - 14 = 0 \quad (1)$$

But the given equation of chord of contact is

$$9x + y - 28 = 0 \quad (2)$$

$\therefore$ Comparing the ratio of coefficients of equations (1) and (2), we get

$$\frac{4h - 3}{9} = \frac{4k + 5}{1} = \frac{3h - 5k + 14}{28}$$

Solving we get $h = 3$ and $k = -1$.

14. Normal passes through the center of the circle.

Hence, the equation of normal is $x - y = 0$.

Exercise 4.5

1. (i) Centres of the given circles are $C_1(0, 1)$, $C_2(1, 0)$ and corresponding radii are $r_1 = 3$, $r_2 = 5$.

$$C_1C_2 = \sqrt{2} < (r_2 - r_1)$$

Therefore, one circle lies entirely inside the other.

Hence, there is no common tangent.

- (ii) Centres of the given circles are $C_1(6, 6)$, $C_2(-3, -3)$ and corresponding radii are $r_1 = 6\sqrt{2}$, $r_2 = 3\sqrt{2}$.

$$C_1C_2 = 9\sqrt{2} = r_1 + r_2$$

Therefore, circles are touching externally.

Hence, there are two common tangents.

2. We have centres $C_1(2, 3)$ and $C_2(5, 6)$.

$$C_1C_2 = 3\sqrt{2}$$

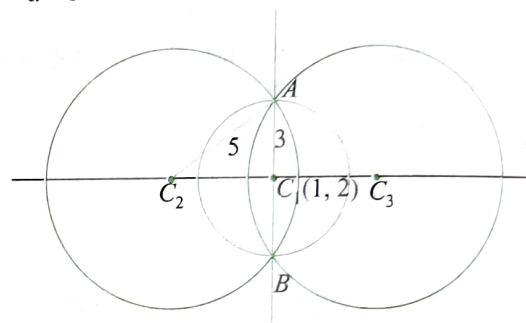
Circles intersect orthogonally.

$$\therefore (C_1C_2)^2 = r_1^2 + r_2^2$$

$$\Rightarrow 18 = 2a^2$$

$$\Rightarrow a = 3$$

- 3.



Clearly, common chord AB will be along the diameter of the given circle.

In triangle AC_1C_2 , $AC_1 = 3$, $AC_2 = 5$.

$$\therefore C_1C_2 = C_1C_3 = 4$$

Given that slope of AB is $\frac{3}{4}$.

$$\therefore \text{Slope of } C_2C_3, -\frac{4}{3} = \tan \theta$$

Using parametric form of straight line, coordinates of C_2 and C_3 are given by

$$\begin{aligned} (1 \pm 4 \cos \theta, 2 \mp 4 \sin \theta) &\equiv \left(1 \pm 4 \frac{3}{5}, 2 \mp 4 \frac{4}{5} \right) \\ &\equiv \left(\frac{17}{5}, \frac{-6}{5} \right) \text{ and } \left(\frac{-7}{5}, \frac{26}{5} \right) \end{aligned}$$

4. Equation of the circle having the end points of diameter as $(1, -3)$ and $(4, 1)$ is

$$(x - 1)(x - 4) + (y + 3)(y - 1) = 0$$

$$\therefore x^2 + y^2 - 5x + 2y + 1 = 0$$

Equation of common chord is $x + y - 1 = 0$.

Equation of family of circles through point of intersection of common chord and circle is

$$x^2 + y^2 - 5x + 2y + 1 + \lambda(x + y - 1) = 0$$

If it passes through $(1, 2)$, then $\lambda = -\frac{5}{2}$.

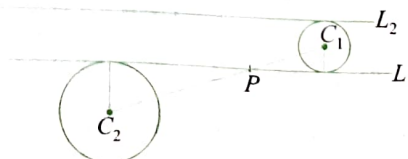
Therefore, the equation of circle is $x^2 + y^2 - \frac{15}{2}x - \frac{y}{2} + \frac{7}{2} = 0$.

5. $S_1 \equiv x^2 + y^2 - 2x - 16y + 64 = 0$

$$S_2 \equiv x^2 + y^2 - 2x + 2y - 2 = 0$$

$$C_1 \equiv (1, 8), r_1 = 1$$

$$C_2 \equiv (1, -1), r_2 = 2$$



Line L_1 is transverse common tangent of S_1 and S_2 .
So, P divides C_1C_2 internally in the ratio of 1 : 2.

$$\Rightarrow P \equiv \left(\frac{1+2}{1+2}, \frac{-1+16}{1+2} \right) \equiv (1, 5)$$

Therefore, equation of L_1 with positive slope is
 $(y-5) = m(x-1)$

$$\text{or } mx - y - m + 5 = 0$$

This is also tangent to S_2 .

So, we have

$$\left| \frac{m+1-m+5}{\sqrt{m^2+1}} \right| = 2$$

$$\therefore m = 2\sqrt{2}$$

So, equation of L_1 is $2\sqrt{2}x - y - 2\sqrt{2} + 5 = 0$.

Thus, equation of L_2 is $2\sqrt{2}x - y + k = 0$.

From the figure, distance between parallel lines is 2.

$$\therefore \left| \frac{k+2\sqrt{2}-5}{3} \right| = 2$$

$$\therefore k = \pm 6 + 5 - 2\sqrt{2}$$

Therefore, equation of L_2 with +ve y -intercept is

$$2\sqrt{2}x - y + 11 - 2\sqrt{2} = 0$$

6. The given circles are

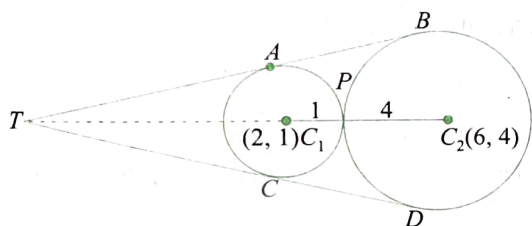
$$x^2 + y^2 - 4x - 2y + 4 = 0 \quad (1)$$

$$\text{and } x^2 + y^2 - 12x - 8y + 36 = 0 \quad (2)$$

with centers $C_1(2, 1)$ and $C_2(6, 4)$ and radii 1 and 4, respectively.

Also, $C_1C_2 = 5$.

As $r_1 + r_2 = C_1C_2$, the two circles touch each other externally at P .



Clearly, P divides C_1C_2 in the ratio 1 : 4.

Therefore, the coordinates of P are

$$\left(\frac{1 \times 6 + 4 \times 2}{1+4}, \frac{1 \times 4 + 4 \times 1}{1+4} \right) \equiv \left(\frac{14}{5}, \frac{8}{5} \right)$$

Let AB and CD be two common tangents of the given circles, meeting each other at T .

Then T divides C_1C_2 externally in the ratio 1 : 4. Hence,

$$T \equiv \left(\frac{1 \times 6 - 4 \times 2}{1-4}, \frac{1 \times 4 - 4 \times 1}{1-4} \right) \equiv \left(\frac{2}{3}, 0 \right)$$

Let m be the slope of the tangent. Then the equation of tangent through $(2/3, 0)$ is

$$y - 0 = m \left(x - \frac{2}{3} \right)$$

$$\text{or } y - mx + \frac{2}{3}m = 0$$

Now, the length of perpendicular from $(2, 1)$ to the above tangent is the radius of the circle. Therefore,

$$\left| \frac{1 - 2m + \frac{2}{3}m}{\sqrt{m^2 + 1}} \right| = 1$$

$$\text{or } (3 - 4m)^2 = 9(m^2 + 1)$$

$$\text{or } 9 - 24m + 16m^2 = 9m^2 + 9$$

$$\text{or } 7m^2 - 24m = 0$$

$$\text{or } m = 0, \frac{24}{7}$$

Thus, the equations of the tangents are $y = 0$ and $7y - 24x + 16 = 0$.

7. Here, $OB = \text{Radius} = 2$

The distance of $(0, 0)$ from $x + y = 5\sqrt{2}$ is 5.

Therefore, the radius of the smallest circle is $(5 - 2)/2 = 3/2$, and $OC = 2 + 3/2 = 7/2$.

The slope of OA is $1 = \tan \theta$.
Therefore,

$$\cos \theta = \frac{1}{\sqrt{2}}, \sin \theta = \frac{1}{\sqrt{2}}$$

$$\therefore C \equiv (0 + OC \cdot \cos \theta, 0 + OC \cdot \sin \theta) \equiv \left(\frac{7}{2\sqrt{2}}, \frac{7}{2\sqrt{2}} \right)$$

8. $A_1B_1 = \sqrt{4+4} = 2\sqrt{2}$

$$\therefore AB = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

Thus, the equation of the required circle is $x^2 + y^2 = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$.

9. The given circle is

$$x^2 + y^2 - 4x - 6y - 12 = 0 \quad (1)$$

whose center is $C_1(2, 3)$ and radius r_1 is $C_1A = 5$.

If $C_2(h, k)$ is the center of the circle of radius 3 which touches the circle (1) internally at $A(-1, -1)$, then $C_2A = 3$ and $C_1C_2 = C_1A - C_2A = 5 - 3 = 2$.

Thus, $C_2(h, k)$ divides C_1A in the ratio 2 : 3 internally. Therefore,

$$h = \frac{2(-1) + 3 \times 2}{2+3} = \frac{4}{5}$$

$$\text{and } k = \frac{2(-1) + 3 \times 3}{2+3} = \frac{7}{5}$$

Hence, the equation of the required circle is $(x - 4/5)^2 + (y - 7/5)^2 = 3^2$, i.e., $5x^2 + 5y^2 - 8x - 14y - 32 = 0$.

10. From $\triangle MLN$,

$$\sin \frac{\theta}{2} = \frac{a-b}{a+b}$$

$$\text{or } \frac{\theta}{2} = \sin^{-1} \left(\frac{a-b}{a+b} \right)$$

Therefore, the angle between AB and AD is

$$\theta = 2 \sin^{-1} \left(\frac{a-b}{a+b} \right)$$

11. The given circles are

$$(x-1)^2 + (y-2)^2 = 1 \quad (1)$$

$$\text{and } (x-7)^2 + (y-10)^2 = 4 \quad (2)$$

Let $A \equiv (1, 2)$, $B \equiv (7, 10)$, $r_1 = 1$, $r_2 = 2$

$$AB \equiv 10, r_1 + r_2 = 3.$$

Now, $AB > r_1 + r_2$. Hence, the two circles are non-intersecting.

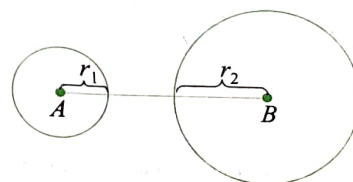
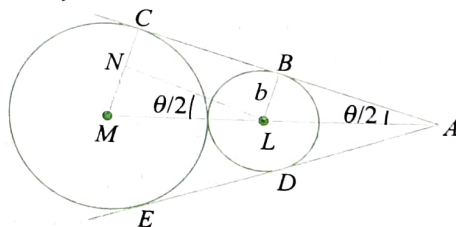
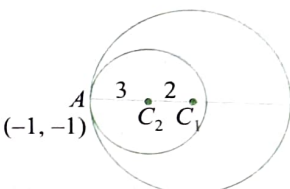
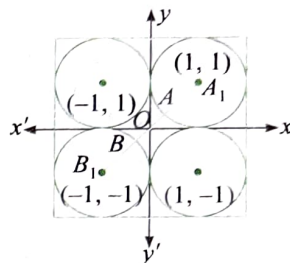
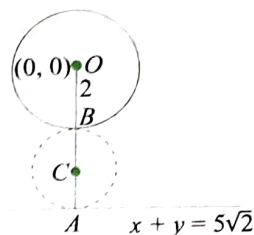
The radii of the two circles at time t are $1 + 0.3t$ and $2 + 0.4t$.

For the two circles to touch each other,

$$AB^2 = [(r_1 + 0.3t) \pm (r_2 + 0.4t)]^2$$

$$\text{or } 100 = [(1 + 0.3t) \pm (2 + 0.4t)]^2$$

$$\text{or } 100 = (3 + 0.7t)^2, [(0.1)t + 1]^2$$



or $3 + 0.7t = \pm 10, 0.1t + 1 = \pm 10$

or $t = 10, t = 90$ [$\because t > 0$]

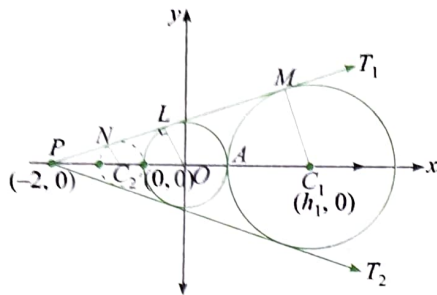
The two circles touch each other externally in 10 s and internally in 90 s.

12. The given circle is

$$x^2 + y^2 = 1 \quad (1)$$

Center = $O(0, 0)$ and radius = 1

Let T_1 and T_2 be the tangents drawn from $(-2, 0)$ to the circle (1).



From the figure, $\triangle PLO$ and $\triangle PMC_1$ are similar. Therefore,

$$\frac{OL}{OP} = \frac{C_1M}{C_1P}$$

or $\frac{1}{2} = \frac{r_1}{h_1 + 2}$

or $2r_1 = h_1 + 2 \quad (2)$

Also, the circles are touching externally. Therefore,

$$h_1 = r_1 + 1 \quad (3)$$

From (2) and (3),

$$r_1 = 3 \text{ and } h_1 = 4$$

Hence, the equation of the circle is

$$(x - 4)^2 + y^2 = 9 \quad (4)$$

Also, $\triangle PLO$ and $\triangle PNC_2$ are similar. Therefore,

$$\frac{OL}{OP} = \frac{C_2N}{C_2P}$$

or $\frac{1}{2} = \frac{r_2}{2 + h_2}$

or $2r_2 = 2 + h_2 \quad (5)$

Also, the circles are touching externally. Therefore,

$$-h_2 = 1 + r_2 \quad (6)$$

From (5) and (6),

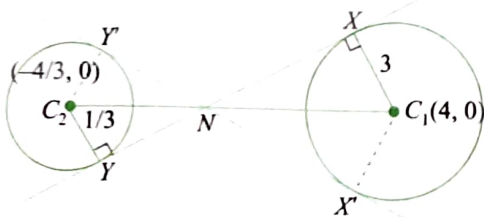
$$r_2 = \frac{1}{3} \text{ and } h_2 = -\frac{4}{3}$$

Hence, the equation of the circle is

$$\left(x + \frac{4}{3}\right)^2 + y^2 = \left(\frac{1}{3}\right)^2 \quad (7)$$

Since circles (1) and (4) are two touching circles, they have three common tangents: T_1 , T_2 , and $x - 1 = 0$.

Similarly, the common tangents of circles (1) and (7) are T_1 , T_2 , and $x = -1$



For the circles (4) and (7), there will be four common tangents: two direct and two indirect.

Two common direct tangents are T_1 and T_2 .

Let us find two common indirect tangents.

$$\frac{C_1N}{C_2N} = \frac{3}{1/3} = 9$$

i.e., N divides C_1C_2 in the ratio 9 : 1. Therefore,

$$N \equiv \left(\frac{9 \times (-4/3) + 1 \times 4}{10}, 0 \right) \equiv \left(-\frac{4}{5}, 0 \right)$$

Any line through N is

$$y = m\left(x + \frac{4}{5}\right)$$

or $5mx - 5y + 4m = 0$

If it is tangent to circle (4), then

$$\left| \frac{20m + 4m}{\sqrt{25m^2 + 25}} \right| = 3$$

or $24m = 15\sqrt{m^2 + 1}$

or $64m^2 = 25m^2 + 25$

or $39m^2 = 25$

or $m = \pm \frac{5}{\sqrt{39}}$

Therefore, the required tangents are

$$y = \pm \frac{5}{\sqrt{39}} \left(x + \frac{4}{5} \right)$$

Exercise 4.6

1. The equation of the common chord AB of the two circles is $2x + 1 = 0$ (using $S_1 - S_2 = 0$).

The equation of family of circles through A and B is

$$(x^2 + y^2 + 2x + 3y + 1) + \lambda(2x + 1) = 0$$

[Using $S_1 + \lambda(S_2 - S_1) = 0$]

or $x^2 + y^2 + 2x(\lambda + 1) + 3y + \lambda + 1 = 0$

Since AB is a diameter of this circle, the center lies on $2x + 1 = 0$. So,

$$-2\lambda - 2 + 1 = 0$$

or $\lambda = -\frac{1}{2}$

Thus, the required circle is

$$x^2 + y^2 + x + 3y + \frac{1}{2} = 0$$

or $2x^2 + 2y^2 + 2x + 6y + 1 = 0$

2. The equation of the circle touching the line $3x - y - 6 = 0$ at $(1, -3)$ is given by

$$(x - 1)^2 + (y + 3)^2 + \lambda(3x - y - 6) = 0$$

i.e., $x^2 + y^2 + (3\lambda - 2)x + (6 - \lambda)y + 10 - 6\lambda = 0 \quad (1)$

Equation (i) touches $y = x$. Therefore,

$$2x^2 + (2\lambda + 4)x + 10 - 6\lambda = 0$$

i.e., $x^2 + (\lambda + 2)x + 5 - 3\lambda = 0 \quad (2)$

has equal roots. Therefore,

$$(\lambda + 2)^2 - 4(5 - 3\lambda) = \lambda^2 + 16\lambda - 16 = 0$$

or $\lambda = \frac{1}{2} [-16 \pm \sqrt{256 + 64}] = -8 \pm \sqrt{80}$

Now, $(\text{Radius})^2 = R^2 = \frac{1}{4} [(3\lambda - 2)^2 + (6 - \lambda)^2 - (10 - 6\lambda)4]$
 $= \frac{1}{4} [10\lambda^2]$

or $R = \frac{\sqrt{10}\lambda}{2} = \frac{\sqrt{10}}{2} (-8 \pm 4\sqrt{5}) = |-4\sqrt{10} \pm 10\sqrt{2}|$

or Radius of smaller circle = $|-4\sqrt{10} + 10\sqrt{2}| = 1.5$ approx.

3. The equation of line AB is

$$y - 2 = \frac{2 - 1}{-2 - 0}(x + 2) = -\frac{1}{2}(x + 2)$$

or $x + 2y - 2 = 0 \quad (1)$

The equation of the circle whose diagonally opposite points are A and B is

$$(x-0)(x+2) + (y-1)(y-2) = 0$$

$$\text{or } x^2 + y^2 + 2x - 3y + 2 = 0$$

The family of circles passing through the points of intersection of (1) and (2) is

$$x^2 + y^2 + 2x - 3y + 2 + \lambda(x+2y-2) = 0$$

$$\text{or } x^2 + y^2 + (2+\lambda)x + (2\lambda-3)y + 2-2\lambda = 0$$

Equation (3) represents a circle of radius $\sqrt{10}$ units. Therefore,

$$\sqrt{\left(-\frac{2+\lambda}{2}\right)^2 + \left(-\frac{2\lambda-3}{2}\right)^2} - 2 + 2\lambda = \sqrt{10}$$

$$\text{or } (4+4\lambda+\lambda^2) + (4\lambda^2+9-12\lambda) + 8\lambda-8 = 40$$

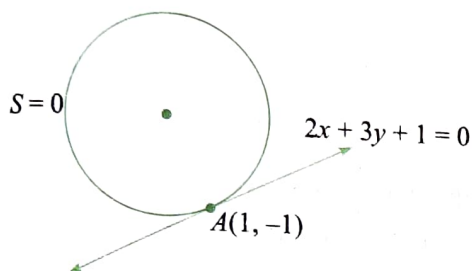
$$\text{or } \lambda = \pm\sqrt{7}$$

Hence, the required circles are

$$x^2 + y^2 + 2x - 3y + 2 \pm \sqrt{7}(x+2y-2) = 0$$

There are two such circles possible.

4. We are given that line $2x + 3y + 1 = 0$ touches the circle $S=0$ at $(1, -1)$.



So, the equation of this circle can be given by

$$(x-1)^2 + (y+1)^2 + \lambda(2x+3y+1) = 0, \lambda \in R$$

[Here, $(x-1)^2 + (y+1)^2 = 0$ represents a point circle at $(1, -1)$.]

$$\text{or } x^2 + y^2 + 2x(\lambda-1) + y(3\lambda+2) + (\lambda+2) = 0 \quad (1)$$

But this circle is orthogonal to the circle the extremities of whose diameter are $(0, 3)$ and $(-2, -1)$, i.e.,

$$x(x+2) + (y-3)(y+1) = 0$$

$$\text{or } x^2 + y^2 + 2x - 2y - 3 = 0 \quad (2)$$

Applying the condition of orthogonality for (1) and (2), we get

$$2(\lambda-1) \times 1 + 2\left(\frac{3\lambda+2}{2}\right) \times (-1) = \lambda+2 + (-3)$$

$$[2g_1g_2 + 2f_1f_2 = c_1 + c_2]$$

$$\text{or } 2\lambda - 2 - 3\lambda - 2 = \lambda - 1$$

$$\text{or } 2\lambda = -3$$

$$\text{or } \lambda = -\frac{3}{2}$$

Substituting this value of λ in (1), we get the required circle as

$$x^2 + y^2 - 5x - \frac{5}{2}y + \frac{1}{2} = 0$$

$$\text{or } 2x^2 + 2y^2 - 10x - 5y + 1 = 0$$

5. Any circle which touches the line $x+y-2=0$ at $(1, 1)$ will be of the form

$$(x-1)^2 + (y-1)^2 + \lambda(x+y-2) = 0$$

$$\text{or } x^2 + y^2 + (\lambda-2)x + (\lambda-2)y + 2-2\lambda = 0$$

The common chord of this circle and $x^2 + y^2 + 4x + 5y - 6 = 0$ will be

$$(\lambda-6)x + (\lambda-7)y + 8-2\lambda = 0$$

$$\text{or } (-6x-7y+8) + \lambda(x+y-2) = 0$$

which is a family of lines, each member of which will be passing through a fixed point, which is the point of intersection of the lines $-6x-7y+8=0$ and $x+y-2=0$, i.e., $(6, -4)$.

Exercises

Single Correct Answer Type

1. (1) If there are more than one rational points on the circumference of the circle $x^2 + y^2 - 2\pi x - 2ey + c = 0$ [as (π, e) is the center], then e will be a rational multiple of π , which is not possible. Thus, the number of rational points on the circumference of the circle is at most one.

2. (1) The two normals are $x=1$ and $y=2$. Their point of intersection $(1, 2)$ is the center of the required circle.

$$\text{Radius} = \frac{|3+8-6|}{5} = 1$$

Therefore, the required circle is

$$(x-1)^2 + (y-2)^2 = 1$$

$$\text{i.e., } x^2 + y^2 - 2x - 4y + 4 = 0$$

3. (2) Let the orthocenter be $H(5, 8)$.

$$\text{Now, } \angle HBM = \frac{\pi}{2} - C$$

$$\text{Also, } \angle DBC = \angle DAC = \frac{\pi}{2} - C$$

Hence, $\triangle BMH$ and $\triangle BMD$ are congruent. Therefore,

$$HM = MD$$

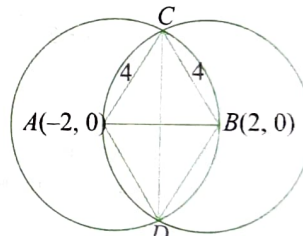
D is the image of H in the line $x-y=0$, which is $D(8, 5)$.

Thus, the equation of the circumcircle is

$$(x-2)^2 + (y-3)^2 = (8-2)^2 + (5-3)^2$$

$$\text{i.e., } x^2 + y^2 - 4x - 6y - 27 = 0$$

4. (1)



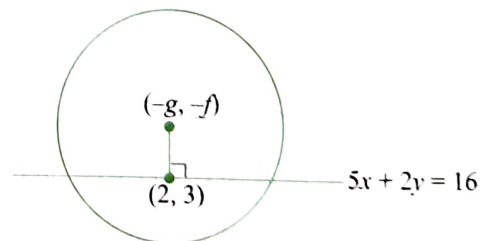
The circles are with centers $(2, 0)$ and $(-2, 0)$ and radius 4.

Therefore, the y -axis is their common chord.

$\triangle ABC$ is equilateral. Hence,

$$\text{Area of } ADBC = \frac{2 \times \sqrt{3}}{4} (4)^2 = 8\sqrt{3}$$

5. (1)



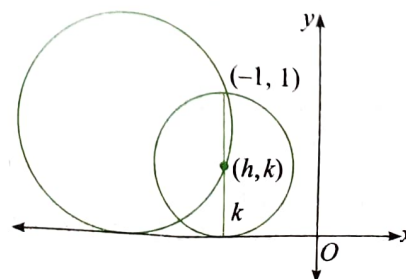
Slope of the given line $= -\frac{5}{2}$

$$\therefore \left(-\frac{5}{2}\right) \left(\frac{3+f}{2+g}\right) = -1$$

$$\text{or } 15 + 5f = 4 + 2g$$

Therefore, the locus is $2x - 5y + 11 = 0$.

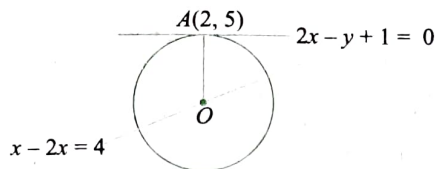
6. (1)



From the figure, $k \geq 1/2$.

7. (1) $2x - y + 1 = 0$ is a tangent.

$$\text{Slope of line } OA = -\frac{1}{2}$$



The equation of OA is

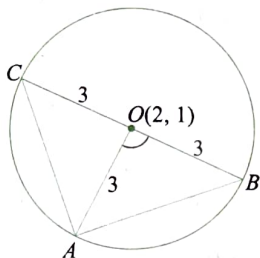
$$(y - 5) = -\frac{1}{2}(x - 2)$$

$$\text{or } x + 2y = 12$$

Therefore, intersection with $x - 2y = 4$ will give the coordinates of center as $(8, 2)$. Hence,

$$r = OA = \sqrt{(8-2)^2 + (2-5)^2} = 3\sqrt{5}$$

8. (1)



From the figure length of side of triangle is $3\sqrt{2}$.

$$9. (4) \quad x^2 + 2ax + c = (x - 2)^2$$

$$\text{or } -2a = 4, c = 4$$

$$\text{or } a = -2, c = 4$$

$$y^2 + 2by + c = (y - 2)(y - 3)$$

$$\text{or } -2b = 5, c = 6$$

$$\text{or } b = -\frac{5}{2}, c = 6$$

Clearly, the data are not consistent.

10. (2) In an equilateral triangle, circumcenter and incenter are coincident. Therefore,

$$\text{Incenter} \equiv (-g, -f)$$

Point $(1, 1)$ lies on the circle. Therefore,

$$1^2 + 1^2 + 2g + 2f + c = 0$$

$$\text{or } c = -2(g + f + 1)$$

Also, in an equilateral triangle,

$$\text{Circumradius} = 2 \times \text{Inradius}$$

$$\therefore \text{Inradius} = \frac{1}{2} \times \sqrt{g^2 + f^2 - c}$$

Therefore, the equation of the incircle is

$$(x + g)^2 + (y + f)^2 = \frac{1}{4}(g^2 + f^2 - c)$$

$$= \frac{1}{4}\{g^2 + f^2 + 2(g + f + 1)\}$$

11. (3) For such triangle, first circle is the incircle and other one is the circumcircle.

Also, the radius of first circle is half of the radius of the second circle.

So, triangle is equilateral.

Thus, incentre and circumcentre coincide.

$$\Rightarrow \alpha - 2\beta = 1 \text{ and } \alpha + \beta = 2$$

$$\Rightarrow (\alpha, \beta) \equiv \frac{5}{3}, \frac{1}{3}$$

$$12. (3) \quad (x \cos \alpha + y \sin \alpha - a)^2 + (x \sin \alpha - y \cos \alpha - b)^2 = k^2$$

$$\text{or } x^2 + y^2 + (-2a \cos \alpha - 2b \sin \alpha)x + (-2a \sin \alpha + 2b \cos \alpha)y + a^2 + b^2 - k^2 = 0$$

Centre of the circle is

$$(a \cos \alpha + b \sin \alpha, a \sin \alpha - b \cos \alpha) \equiv (h, k).$$

$$h \equiv a \cos \alpha + b \sin \alpha$$

$$k \equiv a \sin \alpha - b \cos \alpha$$

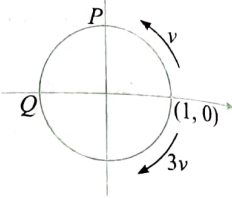
Squaring and adding, we get

$$h^2 + k^2 = a^2 + b^2$$

or $x^2 + y^2 = a^2 + b^2$, which is required locus.

13. (4) The particle which moves clockwise is moving three times as fast as the particle moving anticlockwise. This means the clockwise particle travels $(3/4)$ th of the way around the circle; the anticlockwise particle will travel $(1/4)$ th of the way around the circle. So, the second particle will meet at $P(0, 1)$.

Using the same logic, they will meet at $Q(-1, 0)$ when they meet the second time.



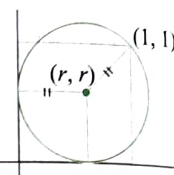
14. (1) From the figure,

$$2(1 - r)^2 = r^2$$

$$\text{or } \sqrt{2}(1 - r) = r$$

$$\text{or } r(\sqrt{2} + 1) = \sqrt{2}$$

$$\text{or } r = \frac{\sqrt{2}}{\sqrt{2} + 1} = \sqrt{2}(\sqrt{2} - 1) = 2 - \sqrt{2}$$



15. (3) The equation of line PQ is

$$y - k = -\frac{h}{k}(x - h)$$

$$\text{or } hx + ky = h^2 + k^2$$

$$\therefore Q \equiv \left(\frac{h^2 + k^2}{h}, 0 \right)$$

$$\text{and } P \equiv \left(0, \frac{h^2 + k^2}{k} \right)$$

$$\text{Also, } 2a = \sqrt{x_1^2 + y_1^2}$$

$$\text{or } x_1^2 + y_1^2 = 4a^2$$

Eliminating x_1 and y_1 , we have

$$(x^2 + y^2)^2 \left(\frac{1}{x^2} + \frac{1}{y^2} \right) = 4a^2$$

16. (1) The equation of the line joining $A(1, 0)$ and $B(3, 4)$ is $y = 2x - 2$.

This cuts the circle $x^2 + y^2 = 4$ at $Q(0, -2)$ and $P(8/5, 6/5)$.

We have $BQ = 3\sqrt{5}$, $QA = \sqrt{5}$, $BP = 7/\sqrt{5}$ and $PA = 3/\sqrt{5}$. Therefore,

$$\alpha = \frac{BP}{PA} = \frac{7/\sqrt{5}}{3/\sqrt{5}} = \frac{7}{3}$$

$$\text{and } \beta = \frac{BQ}{QA} = \frac{3\sqrt{5}}{\sqrt{5}} = 3$$

Therefore, α and β are roots of the equation

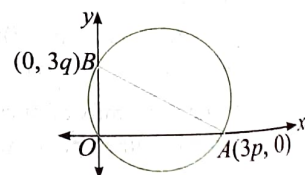
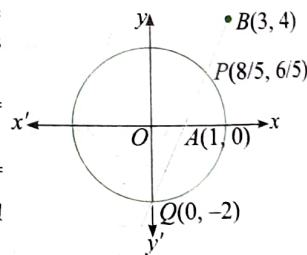
$$x^2 - x(\alpha + \beta) + \alpha\beta = 0$$

$$\text{i.e., } x^2 - x\left(\frac{7}{3} + 3\right) + \frac{7}{3}(3) = 0$$

$$\text{or } 3x^2 - 16x + 21 = 0$$

17. (1) Let the centroid of triangle OAB be (p, q) .

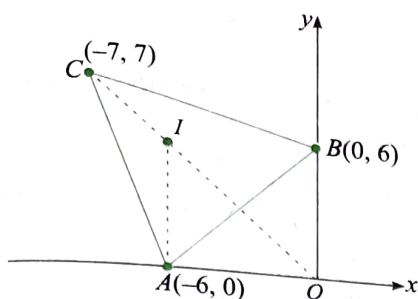
Hence, points A and B are $(3p, 0)$ and $(0, 3q)$, respectively.



But diameter of circle, $AB = 6k$

Hence, $\sqrt{9p^2 + 9q^2} = 6k$

Therefore, the locus of (p, q) is $x^2 + y^2 = 4k^2$.



The triangle is evidently isosceles and, therefore, the median through C is the angle bisector of $\angle C$.

The equation of the angle bisector is $y = -x$ and the incentre $I \equiv (-a, a)$, where a is positive.

The equation of AC is $y - 0 = -7(x + 6)$, i.e., $7x + y + 42 = 0$ and the equation of AB is $x - y + 6 = 0$.

The lengths of the perpendiculars from I to AB and AC are equal. Therefore,

$$\left| \frac{-7a + a + 42}{\sqrt{50}} \right| = \left| \frac{-a - a + 6}{\sqrt{2}} \right|$$

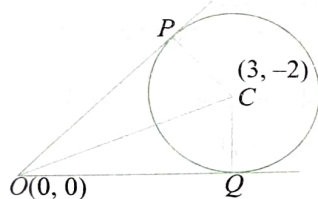
or $a = \frac{9}{2}$ ($\because a > 0$)

Therefore, the center is $(-9/2, 9/2)$ and the radius is $3/\sqrt{2}$.

Hence, the equation of the circle is

$$\left(x + \frac{9}{2}\right)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{9}{2}$$

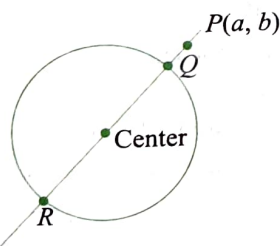
or $x^2 + y^2 + 9x - 9y + 36 = 0$



Clearly, $OPCQ$ is a cyclic quadrilateral. Then the circumcircle of ΔOPQ passes through the point C .

For this circle, OC is diameter. Then the center is the midpoint of OC , which is $(3/2, -1)$.

20. (1) The given circle is $(x + 1)^2 + (y + 2)^2 = 9$, which has radius 3.



The points on the circle which are nearest and farthest to the point $P(a, b)$ are Q and R , respectively.

Thus, the circle centered at Q having radius PQ will be the smallest circle while the circle centered at R having radius PR will be the largest required circle.

Hence, the difference between their radii is $PR - PQ = QR = 6$.

21. (1) The curve passing through the point of intersection of S and S' is

$$S + \lambda S' = 0$$

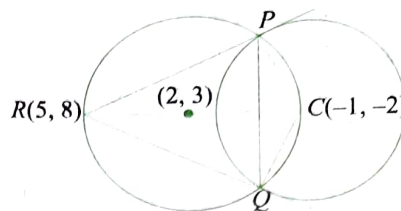
or $x^2(\sin^2\theta + \lambda \cos^2\theta) + y^2(\cos^2\theta + \lambda \sin^2\theta) + 2xy(h + \lambda h')$
 $+ x(32 + 16\lambda) + y(16 + 32\lambda) + 19(1 + \lambda) = 0$

For this equation to be a circle,

$$\sin^2\theta + \lambda \cos^2\theta = \cos^2\theta + \lambda \sin^2\theta \text{ or } \lambda = 1$$

and $h + \lambda h' = 0$ or $h + h' = 0$

22. (1)



Let C be the center of the given circle.

Then, the circumcircle of ΔRPQ passes through C .

Therefore, $(2, 3)$ is the midpoint of RC .

Thus, the coordinates of C are $(-1, -2)$.

Hence, the equation of the circle is $x^2 + y^2 + 2x + 4y - 20 = 0$.

23. (2) If (α, β) is the center, then

$$(\alpha - 1)^2 + (\beta - 3)^2 = (\alpha - 3)^2 + (\beta - 1)^2 \quad (1)$$

and $\frac{\beta - 3}{\alpha - 1} \cdot \frac{\beta - 1}{\alpha - 3} = -1$

or $(\alpha - 1)(\alpha - 3) + (\beta - 1)(\beta - 3) = 0 \quad (2)$

From (1),

$$4\alpha - 4\beta = 0$$

$\therefore \alpha = \beta$

From (2),

$$2(\alpha - 1)(\alpha - 3) = 0$$

$\therefore \alpha = 1, 3$

$\therefore (\alpha, \beta) \equiv (1, 1), (3, 3)$

24. (4) Any point on the line $7x + y + 3 = 0$ is $Q(t, -3 - 7t)$, $t \in \mathbb{R}$.

Now, $P(h, k)$ is the image of point Q in the line $x - y + 1 = 0$.

Then,

$$\frac{h - t}{1} = \frac{k - (-3 - 7t)}{-1}$$

$$= -\frac{2\{t - (-3 - 7t) + 1\}}{1 + 1}$$

$$= -8t - 4$$

or $(h, k) \equiv (-7t - 4, t + 1)$

This point lies on the circle $x^2 + y^2 = 9$. Then,

$$(-7t - 4)^2 + (t + 1)^2 = 9$$

or $50t^2 + 58t + 8 = 0$

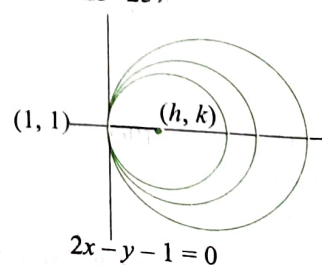
or $25t^2 + 29t + 4 = 0$

or $(25t + 4)(t + 1) = 0$

or $t = \frac{-4}{25}, t = -1$

i.e., $(h, k) \equiv \left(-\frac{72}{25}, \frac{21}{25}\right)$ or $(3, 0)$

25. (2)



Obviously, the locus of the center is a line perpendicular to the given line.

Hence, the locus is

$$\frac{k-1}{h-1} = -\frac{1}{2} \text{ or } x+2y=3$$

26. (4) Let the circle be $x^2 + y^2 = 1$.

Let the vertices of the triangle be $(\cos \theta_i, \sin \theta_i)$, $i = 1, 2, 3$.

Then, the orthocenter is

$$(\cos \theta_1 + \cos \theta_2 + \cos \theta_3, \sin \theta_1 + \sin \theta_2 + \sin \theta_3)$$

Therefore, the distance between the orthocenter and the circumcenter is

$$\begin{aligned} & \sqrt{(\cos \theta_1 + \cos \theta_2 + \cos \theta_3)^2 + (\sin \theta_1 + \sin \theta_2 + \sin \theta_3)^2} \\ &= \sqrt{(\cos^2 \theta_1 + \sin^2 \theta_1) + (\cos^2 \theta_2 + \sin^2 \theta_2) + (\cos^2 \theta_3 + \sin^2 \theta_3) \\ & \quad + 2[\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)]} \\ &= \sqrt{3 + 2[\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)]} < 3 \end{aligned}$$

27. (2) Let $y = mx$ be a chord.

Then the points of intersections are given by

$$x^2(1+m^2) - x(3+4m) - 4 = 0$$

$$\therefore x_1 + x_2 = \frac{3+4m}{1+m^2} \text{ and } x_1 x_2 = \frac{-4}{1+m^2}$$

Since $(0, 0)$ divides chord in the ratio 1 : 4, we have

$$x_2 = -4x_1$$

$$\therefore -3x_1 = \frac{3+4m}{1+m^2} \text{ and } 4x_1 = -\frac{-4}{1+m^2}$$

$$\therefore 9 + 9m^2 = 9 + 16m^2 + 24m$$

$$\text{i.e., } m = 0, -\frac{24}{7}$$

Therefore, the lines are $y = 0$ and $y + 24x = 0$.

28. (3) Let the equation of the chord OA of the circle

$$x^2 + y^2 - 2x + 4y = 0 \quad (1)$$

$$\text{be } y = mx. \quad (2)$$

Solving (1) and (2), we get

$$x^2 + m^2 x^2 - 2x + 4mx = 0$$

$$\text{or } (1+m^2)x^2 - (2-4m)x = 0$$

$$\text{or } x = 0 \text{ and } x = \frac{2-4m}{1+m^2}$$

Hence, the points of intersection are

$$O(0, 0) \text{ and } A\left(\frac{2-4m}{1+m^2}, \frac{m(2-4m)}{1+m^2}\right)$$

$$\text{or } OA^2 = \left(\frac{2-4m}{1+m^2}\right)^2 (1+m^2) = \frac{(2-4m)^2}{1+m^2}$$

Since OAB is an isosceles right-angled triangle,

$$OA^2 = \frac{1}{2} AB^2$$

where AB is a diameter of the given circle. Hence,

$$OA^2 = 10$$

$$\text{or } \frac{(2-4m)^2}{1+m^2} = 10$$

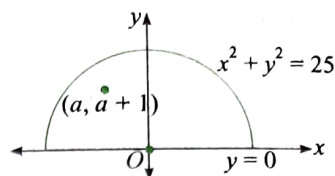
$$\text{or } 4 - 16m + 16m^2 = 10 + 10m^2$$

$$\text{or } 3m^2 - 8m - 3 = 0$$

$$\text{i.e., } m = 3 \text{ or } -\frac{1}{3}$$

Hence, the required equations are $y = 3x$ or $x + 3y = 0$.

29. (3)



$y = \sqrt{25 - x^2}$ and $y = 0$ bound the semicircle above the x -axis.

Therefore,

$$a+1 > 0 \quad (1)$$

$$\text{and } a^2 + (a+1)^2 - 25 < 0 \text{ or } 2a^2 + 2a - 24 < 0$$

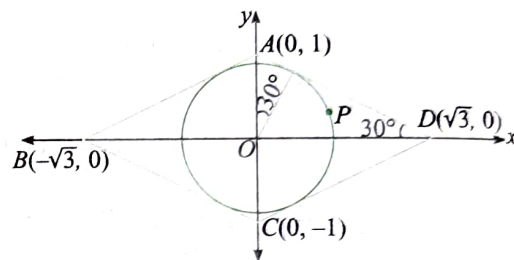
$$\text{or } a^2 + a - 12 < 0$$

$$\text{or } -4 < a < 3 \quad (2)$$

From (1) and (2),

$$-1 < a < 3$$

30. (2)



$$OA = 1$$

$$r = OA \cos 30^\circ = \frac{\sqrt{3}}{2}$$

The equation of circle is

$$x^2 + y^2 = \frac{3}{4}$$

Let $P \equiv (x_1, y_1)$.

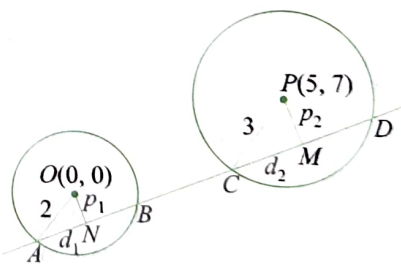
$$PA^2 + PB^2 + PC^2 + PD^2$$

$$= x_1^2 + (y_1 - 1)^2 + (x_1 + \sqrt{3})^2 + y_1^2 + x_1^2 + (y_1 + 1)^2 + (x_1 - \sqrt{3})^2 + y_1^2$$

$$= 4x_1^2 + 4y_1^2 + 8 = 4(x_1^2 + y_1^2) + 8$$

$$= 4 \times \frac{3}{4} + 8 = 11$$

31. (1)



Let the equation of line be

$$y = x + c \text{ or } y - x = c$$

The perpendicular from $(0, 0)$ on line (1) is $|c/\sqrt{2}|$.

The perpendicular from $(5, 7)$ on line (1) is $|c - 2/\sqrt{2}|$.

In $\triangle AON$,

$$\sqrt{2^2 - \left(\frac{c}{\sqrt{2}}\right)^2} = AN$$

and in $\triangle CPM$,

$$\sqrt{3^2 - \left(\frac{2-c}{\sqrt{2}}\right)^2} = CM$$

Given $AN = CM$

$$\text{or } 4 - \frac{c^2}{2} = 9 - \frac{(2-c)^2}{2} \text{ or } c = -\frac{3}{2}$$

Therefore, the equation of the line is
 $y = x - \frac{3}{2}$ or $2x - 2y - 3 = 0$

- (3) The given circle is $x^2 + y^2 = 1$.
 Center, $C \equiv (0, 0)$, radius = 1,
 and chord is $y = mx + 1$.

$$\cos 45^\circ = \frac{CP}{CR}$$

perpendicular distance from
 $(0, 0)$ to the chord $y = mx + 1$,

$$CP = \frac{1}{\sqrt{m^2 + 1}}$$

$$\text{or } \cos 45^\circ = \frac{1/\sqrt{m^2 + 1}}{1}$$

$$\text{or } \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{m^2 + 1}}$$

$$\text{or } m^2 + 1 = 2$$

$$\text{or } m = \pm 1$$

- (3) Slope of $l_1 = \frac{1}{2}$

Slope of $l_2 = -2$

The equation of l_2 is

$$y = -2(x - 10)$$

$$\text{or } y + 2x = 20$$

Hence, $t = 20$

- (4) $2PQ = PA + PB$

$$\text{or } PQ - PA = PB - PQ$$

$$\text{or } AQ = QB$$

Therefore, Q is the
 midpoint of AB .

Let Q has coordinates
 (h, k) .

Then the equation of chord AB is given by

$$T = S_1$$

$$\text{or } hx + ky - 4 = h^2 + k^2 - 4$$

This variable chord passes through the point $P(3, 5)$. Therefore,

$$3h + 5k = h^2 + k^2$$

$$\text{or } x^2 + y^2 - 3x - 5y = 0$$

which is the required locus.

- (2) The given point is an interior point. Therefore,

$$\left(-5 + \frac{r}{\sqrt{2}}\right)^2 + \left(-3 + \frac{r}{\sqrt{2}}\right)^2 - 16 < 0$$

$$\text{or } r^2 - 8\sqrt{2}r + 18 < 0$$

$$\text{or } 4\sqrt{2} - \sqrt{14} < r < 4\sqrt{2} + \sqrt{14} \quad (1)$$

The point is in the major segment. Therefore, the center and the
 point are on the same side of the line $x + y = 2$. Thus,

$$-5 + \frac{r}{\sqrt{2}} - 3 + \frac{r}{\sqrt{2}} - 2 < 0$$

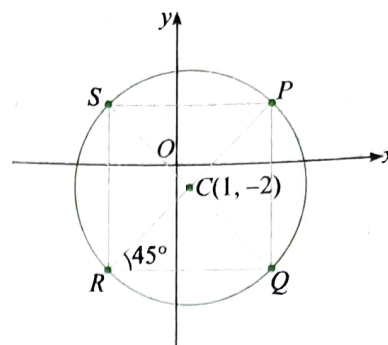
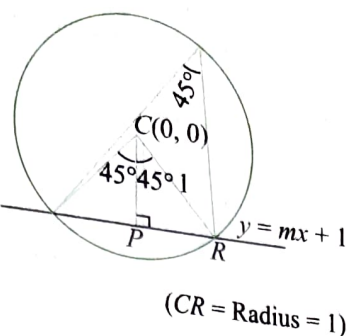
$$\text{or } r < 5\sqrt{2} \quad (2)$$

From (1) and (2), $4\sqrt{2} - \sqrt{14} < r < 5\sqrt{2}$.

- (1) Equation of circle is $(x - 1)^2 + (y + 2)^2 = 98$.

Square is inscribed in the given circle such that its sides are
 parallel to coordinate axes.

So, slope of PR is 1 and that of SQ is -1 .



Thus, vertices of square are

$$(1 \pm 7\sqrt{2} \cos 45^\circ, -2 \pm 7\sqrt{2} \sin 45^\circ) \equiv (8, 5) \text{ and } (-6, -9)$$

$$\text{and } (1 \pm 7\sqrt{2} \cos 135^\circ, -2 \pm 7\sqrt{2} \sin 135^\circ) \equiv (8, -9) \text{ and } (-6, 5)$$

37. (1) Any point on the line passing through the point $P(1, -2)$ and
 having slope $\tan \theta$ is

$$(1 + r \cos \theta, -2 + r \sin \theta)$$

It lies on the given circle.

$$\therefore (1 + r \cos \theta)^2 + (-2 + r \sin \theta)^2$$

$$= (1 + r \cos \theta) - (-2 + r \sin \theta) = 0$$

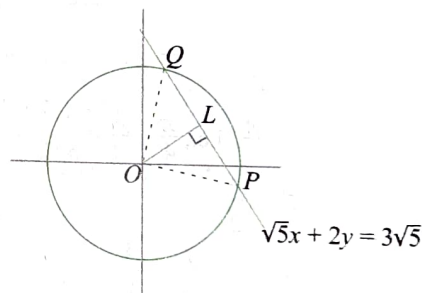
$$\therefore r^2 + (\cos \theta - 5 \sin \theta)r + 6 = 0$$

Above equation has roots r_1 and r_2 , where $r_1 = PA$ and $r_2 = PB$.

$$\therefore PA + PB = r_1 + r_2 = \cos \theta - 5 \sin \theta$$

$$(\cos \theta - 5 \sin \theta)_{\max} = \sqrt{26}$$

38. (3)



The length of perpendicular from the origin to the line
 $x\sqrt{5} + 2y = 3\sqrt{5}$ is

$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$

Radius of the given circle = $\sqrt{10} = OQ = OP$

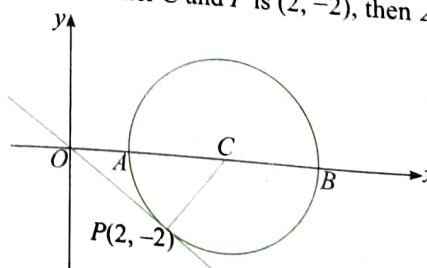
$$PQ = 2QL = 2\sqrt{OQ^2 - OL^2}$$

$$= 2\sqrt{10 - 5} = 2\sqrt{5}$$

$$\text{Thus, area of } \triangle OPQ = \frac{1}{2} \times PQ \times OL$$

$$= \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5} = 5$$

39. (3) If $(a, 0)$ is the center C and P is $(2, -2)$, then $\angle COP = 45^\circ$.



Since the equation of OP is $x + y = 0$, we have

$$OP = 2\sqrt{2} = CP$$

Hence, $OC = 4$.

The point on the circle with the greatest x coordinate is B .

$$\alpha = OB = OC + CB = 4 + 2\sqrt{2}$$

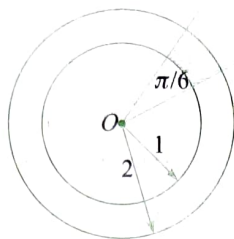
40. (4) The angle θ between the lines represented by

$$\sqrt{3}x^2 - 4xy + \sqrt{3}y^2 = 0 \text{ is given by}$$

$$\begin{aligned} \theta &= \tan^{-1} \frac{\sqrt{h^2 - ab}}{|a + b|} \\ &= \tan^{-1} \frac{2\sqrt{2^2 - 3}}{\sqrt{3} + \sqrt{3}} = \frac{\pi}{6} \end{aligned}$$

Hence, the shaded area is

$$\frac{\pi/6}{2\pi} \times \pi(2^2 - 1^2) = \frac{\pi}{4}$$



41. (2) The slope of the chord is $m = -8/y$. Therefore,

$$y = \pm 1, \pm 2, \pm 4, \pm 8$$

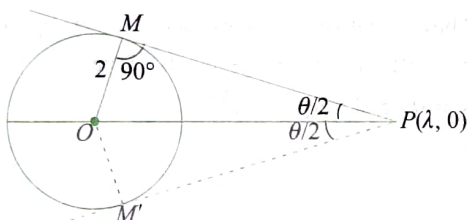
But $(8, y)$ must also lie inside the circle $x^2 + y^2 = 125$.

So, y can be equal to $\pm 1, \pm 2, \pm 4$, i.e., 6 values.

42. (3) The center of the circle is $(1, 0)$ and the radius is 1. The line will touch the circle if $|\cos \theta - 2| = 1$.

So, $\cos \theta = 1$ or $\theta = 2n\pi, n \in \mathbb{I}$.

43. (1)



We have $\frac{\pi}{2} < \theta < \frac{2\pi}{3}$, i.e., $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{3}$

$$\text{or } \frac{1}{\sqrt{2}} < \sin \frac{\theta}{2} < \frac{\sqrt{3}}{2}$$

$$\text{But } \sin \frac{\theta}{2} = \frac{2}{\lambda} \text{ or } \frac{1}{\sqrt{2}} < \frac{2}{\lambda} < \frac{\sqrt{3}}{2}$$

$$\text{or } \frac{4}{\sqrt{3}} < \lambda < 2\sqrt{2}$$

44. (1) Let $A \equiv (1, 0)$, $B \equiv (3, 0)$, and C_1 and C_2 be the centers of circles passing through A and B and touching the y -axis at P_1 and P_2 , respectively. If r is the radius (here radii of both circles will be the same),

$$C_1A = C_2A = r = OD = 2$$

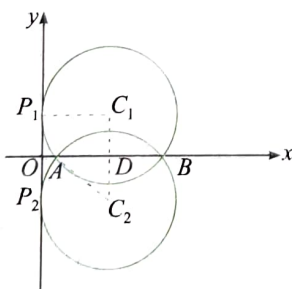
and $C_1 \equiv (2, h)$

$$\text{where } h^2 = AC_1^2 - AD^2 = 4 - 1 = 3$$

$$\text{or } C_1 \equiv (2, \sqrt{3}), C_2 \equiv (2, -\sqrt{3})$$

If $\angle C_1AC_2 = \theta$, then

$$\cos \theta = \frac{AC_1^2 + AC_2^2 - C_1C_2^2}{2AC_1 \times AC_2} = \frac{1}{2}$$



45. (3) Chord with midpoint (h, k) is

$$hx + ky = h^2 + k^2$$

Chord of contact of (x_1, y_1) is

$$xx_1 + yy_1 = 2$$

Comparing (1) and (2), we get

$$x_1 = \frac{2h}{h^2 + k^2} \text{ and } y_1 = \frac{2k}{h^2 + k^2}$$

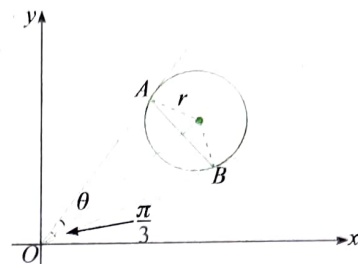
Since (x_1, y_1) lies on $3x + 4y = 10$, $6h + 8k = 10(h^2 + k^2)$.

Therefore, the locus of (h, k) is

$$x^2 + y^2 - \frac{3}{5}x - \frac{4}{5}y = 0$$

which is a circle with center $P(3/10, 4/10)$. Therefore, $OP = 1/2$.

46. (1)



$$\text{Here, } \tan 2\theta = \frac{2\sqrt{4-1}}{2} = \sqrt{3}$$

$$\text{or } \theta = \frac{\pi}{6}$$

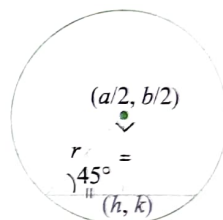
$$\text{Area of } \triangle OAB = \frac{1}{2} (r \cot \theta)^2 (\sin 2\theta) = \frac{1}{2} (r\sqrt{3})^2 \frac{\sqrt{3}}{2}$$

$$47. (3) r = \sqrt{\frac{a^2}{4} + \frac{b^2}{4}} = \frac{\sqrt{a^2 + b^2}}{2}$$

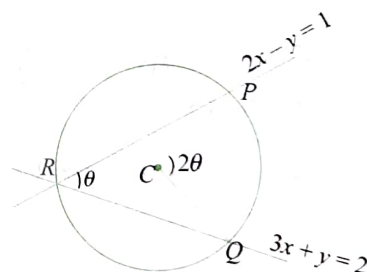
$$\sin 45^\circ = \frac{\sqrt{\left(h - \frac{a}{2}\right)^2 + \left(h - \frac{b}{2}\right)^2}}{\sqrt{a^2 + b^2}/2}$$

$$\text{or } \frac{1}{2} = 4 \left[\frac{(2h-a)^2 + (2h-b)^2}{4(a^2 + b^2)} \right]$$

$$\text{Simplify to get locus } x^2 + y^2 - ax - by - \frac{a^2 + b^2}{8} = 0.$$



48. (2)



Slopes of given lines are 2 and -3.

If angle between lines is θ , then

$$\tan \theta = \left| \frac{2 - (-3)}{1 + (2) \times (-3)} \right| = 1$$

$$\therefore \theta = \angle PRQ = 45^\circ$$

Therefore, angle subtended by chord PQ at centre is 90° .

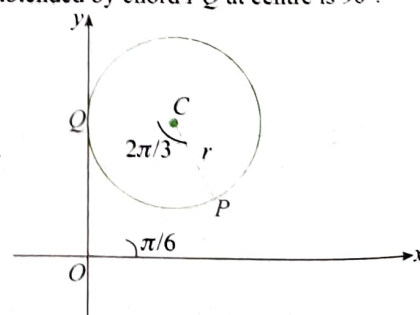
$$49. (2) \tan 30^\circ = \frac{r}{OP}$$

$$\text{or } OP = r\sqrt{3}$$

$$\text{Also, } r^2 \frac{1}{2} \frac{2\pi}{3} = 3\pi$$

$$\therefore r = 3$$

$$\therefore OP = 3\sqrt{3}$$



50. (1) The distance of the given line from the center of the circle is $|p|$.

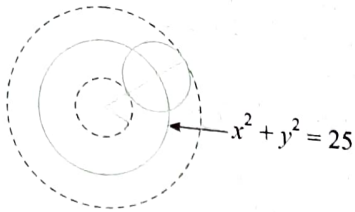
Now, the line subtends right angle at the center. Hence,

$$\text{Radius} = \sqrt{2} |p|$$

$$\text{or } a = \sqrt{2} |p|$$

$$\text{or } a^2 = 2p^2$$

51. (1)



Let (h, k) be any point in the set. Then the equation of circle is $(x - h)^2 + (y - k)^2 = 9$

But (h, k) lies on $x^2 + y^2 = 25$. Therefore, $h^2 + k^2 = 25$

$\therefore 2 \leq \text{Distance between the centers of two circles} \leq 8$
or $4 \leq h^2 + k^2 \leq 64$

Therefore, the locus of (h, k) is $4 \leq x^2 + y^2 \leq 64$.

52. (1) Solving $y = mx$ and $x^2 + y^2 - 2x - 2y = 0$, we get

$$x^2 + m^2x^2 - 2x - 2mx = 0$$

$$\text{or } x = 0, \frac{2(1+m)}{1+m^2}$$

Similarly, solving $y = -mx$ and the equation of the circle, we get

$$x = 0, \frac{2(1-m)}{1+m^2}$$

$$\therefore A \equiv \left(\frac{2(1+m)}{1+m^2}, \frac{2m(1+m)}{1+m^2} \right)$$

$$\text{and } B \equiv \left(\frac{2(1-m)}{1+m^2}, \frac{-2m(1-m)}{1+m^2} \right)$$

Let the middle point of AB be (α, β) . Then

$$\alpha = \frac{1}{2} \left\{ \frac{2(1+m)}{1+m^2} + \frac{2(1-m)}{1+m^2} \right\}$$

$$\text{and } \beta = \frac{1}{2} \left\{ \frac{2m(1+m)}{1+m^2} + \frac{-2m(1-m)}{1+m^2} \right\}$$

$$\therefore \alpha = \frac{2}{1+m^2}, \beta = \frac{2m^2}{1+m^2}$$

Eliminating m from these, we get $\alpha + \beta = 2$.

Hence, the locus is $x + y = 2$.

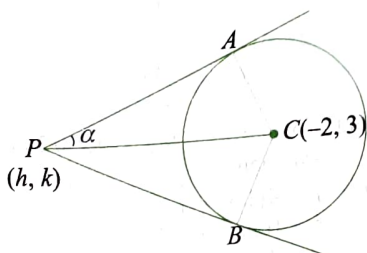
53. (4) The center of the circle

$$x^2 + y^2 + 4x - 6y + 9 \sin^2 \alpha + 13 \cos^2 \alpha = 0$$

is $C(-2, 3)$ and its radius is

$$\sqrt{2^2 + (-3)^2 - 9 \sin^2 \alpha - 13 \cos^2 \alpha}$$

$$\sqrt{4 + 9 - 9 \sin^2 \alpha - 13 \cos^2 \alpha} = |2 \sin \alpha|$$



Let $P(h, k)$ be any point on the locus. Then $\angle APC = \alpha$.

From the diagram,

$$\sin \alpha = \frac{AC}{PC} = \frac{2 \sin \alpha}{\sqrt{(h+2)^2 + (k-3)^2}}$$

$$\text{or } (h+2)^2 + (k-3)^2 = 4$$

$$\text{or } h^2 + k^2 + 4h - 6k + 9 = 0$$

Thus, the required equation of the locus is

$$x^2 + y^2 + 4x - 6y + 9 = 0$$

54. (4) Let $B(h, 0)$ be the midpoint of the chord drawn from point $A(p, q)$.

Also, the center is $C(p/2, q/2)$.

Then, we have $BC \perp AB$. Therefore,

$$\frac{(q/2) - 0}{(p/2) - h} \cdot \frac{(q - 0)}{(p - h)} = -1$$

$$\therefore \left(\frac{q}{p - 2h} \right) \left(\frac{q - 0}{p - h} \right) = -1$$

$$\therefore 2h^2 - 3ph + p^2 + q^2 = 0$$

Since two such chords exist, the above equation must have two distinct real roots, i.e.,

$$\text{Discriminant} > 0$$

$$\therefore 9p^2 - 8(p^2 + q^2) > 0$$

$$\text{or } p^2 > 8q^2$$

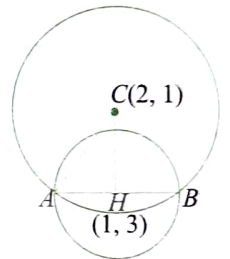
55. (3) The given circle is $x^2 + y^2 - 2x - 6y + 6 = 0$.

Its center is $H(1, 3)$ and radius is 2.

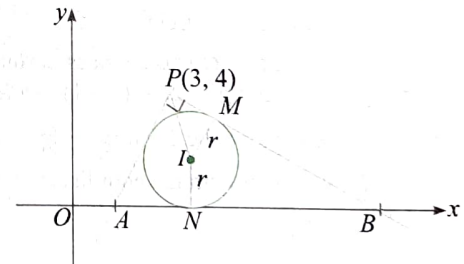
$$AH = 2$$

Radius of the required circle = AC

$$= \sqrt{AH^2 + CH^2} = \sqrt{2^2 + 5} = 3$$



56. (1)



Let point I be (h, k) .

From the figure,

$$IP = r \operatorname{cosec} 45^\circ, \text{ where } r = \text{inradius}$$

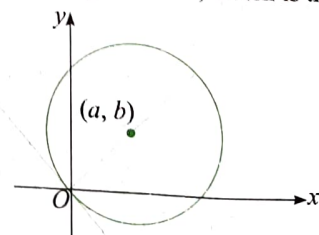
$$\Rightarrow (IP)^2 = 2r^2$$

$$\Rightarrow (3 - h)^2 + (4 - k)^2 = 2(|k|)^2$$

(As circle touches x -axis, $r = |k|$)

$$\Rightarrow x^2 - y^2 - 6x - 8y + 25 = 0, \text{ which is the required locus.}$$

57. (2)



Obviously, the slope of the tangent will be $-\left(\frac{1}{b/a}\right)$, i.e., $-\frac{a}{b}$.

Hence, the equation of the tangent is $y = -\frac{a}{b}x$, i.e., $by + ax = 0$.

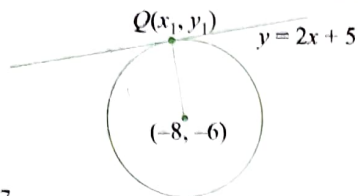
58. (4) From the figure, we have

$$y_1 = 2x_1 + 5$$

$$\text{and } \frac{y_1 + 6}{x_1 + 8} \times 2 = -1$$

$$x_1 + 2y_1 + 20 = 0$$

$$\therefore x_1 = -6 \text{ and } y_1 = -7$$



59. (4) The locus is the radical axis which is perpendicular to the line joining the centers of the circles.

60. (3) The line $5x - 2y + 6 = 0$ is intersected by a tangent at P to the circle $x^2 + y^2 + 6x + 6y - 2 = 0$ on the y -axis at $Q(0, 3)$.

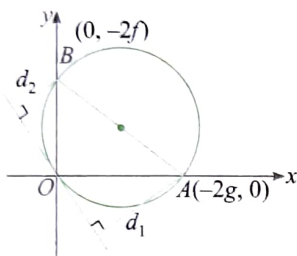
In other words, the tangent passes through $(0, 3)$. Therefore,

$$PQ = \text{Length of tangent to circle from } (0, 3)$$

$$= \sqrt{0 + 9 + 0 + 18 - 2}$$

$$= \sqrt{25} = 5$$

61. (3) Let the circle be $x^2 + y^2 + 2gx + 2fy = 0$.



Tangent at the origin is

$$gx + fy = 0.$$

$$d_1 = \frac{2g^2}{\sqrt{g^2 + f^2}} \text{ and } d_2 = \frac{2f^2}{\sqrt{g^2 + f^2}}$$

$$\text{or } d_1 + d_2 = 2\sqrt{g^2 + f^2}$$

= Diameter of the circle

62. (2) The line $2y = gx + \alpha$ should pass through $(-g, -g)$. So, $-2g = -g^2 + \alpha$ or $\alpha = g^2 - 2g = (g - 1)^2 - 1 \geq -1$

63. (2) Let the tangent be of the form $\frac{x}{x_1} + \frac{y}{y_1} = 1$ and the area of triangle formed by it with the coordinate axes be

$$\frac{1}{2} x_1 y_1 = a^2 \quad (1)$$

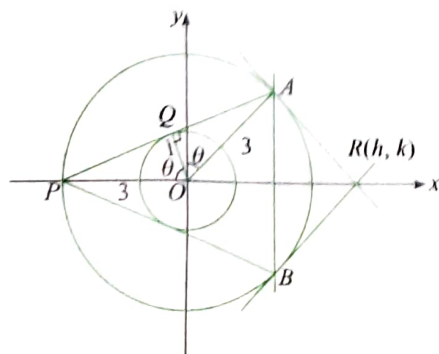
Using the conditions of tangency, we get

$$\frac{|-x_1 y_1|}{\sqrt{x_1^2 + y_1^2}} = a \text{ or } x_1^2 + y_1^2 = \frac{x_1^2 y_1^2}{a^2} \quad (2)$$

From (1) and (2), we get the value of x_1 and y_1 , which gives the equation of tangent as $x \pm y = \pm a\sqrt{2}$.

64. (1) Since $\triangle POQ$ and $\triangle AOQ$ are congruent,

$$\angle POQ = \angle QOA = \theta$$



$$\cos \theta = \frac{1}{3}, \text{ since } \angle POR = \pi$$

$$\therefore \angle AOR = \pi - 2\theta$$

Now, in triangle AOR , $\angle AOR = \pi - 2\theta$ and $AO = 3$ units. Therefore,

$$\cos(\pi - 2\theta) = \frac{OA}{OR} = \frac{3}{\sqrt{h^2 + k^2}}$$

$$\text{or } \sqrt{h^2 + k^2} = \frac{27}{7}$$

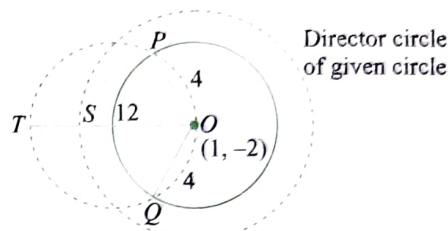
$$\text{or } x^2 + y^2 = \left(\frac{27}{7}\right)^2$$

65. (4) Given circle $(x - 1)^2 + (y + 2)^2 = 16$

Its director circle is

$$(x - 1)^2 + (y + 2)^2 = 32$$

$$\therefore OS = 4\sqrt{2}$$



Therefore, required distance,

$$TS = OT - SO = 12 - 4\sqrt{2}$$

66. (3) Let two given points be $A(a, 0)$ and $B(-a, 0)$ and let the straight line be $y = mx + c$. Then,

$$\frac{mx + c}{\sqrt{1 + m^2}} + \frac{-mx + c}{\sqrt{1 + m^2}} = 2k$$

$$\text{or } c = k\sqrt{1 + m^2}$$

So, the straight line is $y = mx + k\sqrt{1 + m^2}$.

Clearly, it touches the circle $x^2 + y^2 = k^2$ of radius k .

67. (2) The center of $x^2 + y^2 - 4x - 4y = 0$ is $(2, 2)$.

Given normal is $ax + by = 2$. Therefore,

$$2a + 2b = 2 \text{ or } a + b = 1$$

$ax + by = 2$ touches $x^2 + y^2 = 1$. So,

$$1 = \left| \frac{-2}{\sqrt{a^2 + b^2}} \right|$$

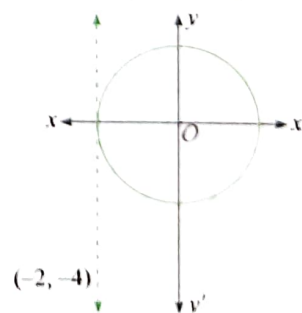
$$\therefore a^2 + b^2 = 4 \text{ or } a^2 + (1 - a)^2 = 4$$

$$\text{or } 2a^2 - 2a - 3 = 0$$

$$\therefore a = \frac{2 \pm \sqrt{4 + 24}}{4} = \frac{1 \pm \sqrt{7}}{2}$$

$$\therefore b = 1 - a = 1 - \frac{1 \pm \sqrt{7}}{2} = \frac{1 \mp \sqrt{7}}{2}$$

68. (1)



Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{1 + m^2}$. If it passes through $(-2, -4)$, then

$$(2m - 4)^2 = 4(1 + m^2)$$

$$\text{or } 4m^2 + 16 - 16m = 4 + 4m^2$$

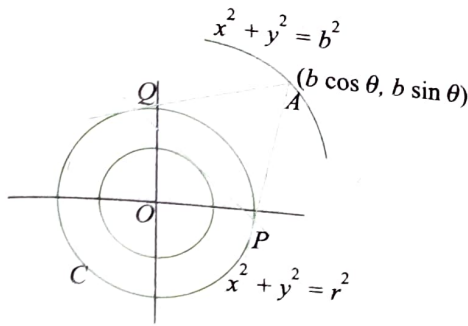
$$\text{or } m = \infty, m = \frac{3}{4}$$

Hence, the slope of the reflected ray is $3/4$.

Thus, the equation of the incident ray is

$$(y+4) = -\frac{3}{4}(x+2) \text{ i.e., } 4y+3x+22=0$$

69. (1)



Chord of contact of point A w.r.t. $x^2 + y^2 = r^2$ is
 $xb \cos \theta + yb \sin \theta = r^2$

This must be a tangent to the circle $x^2 + y^2 = a^2$. Therefore, (1)

$$\left[\frac{r^2}{\sqrt{b^2 \cos^2 \theta + b^2 \sin^2 \theta}} \right] = a \text{ or } r^2 = ab$$

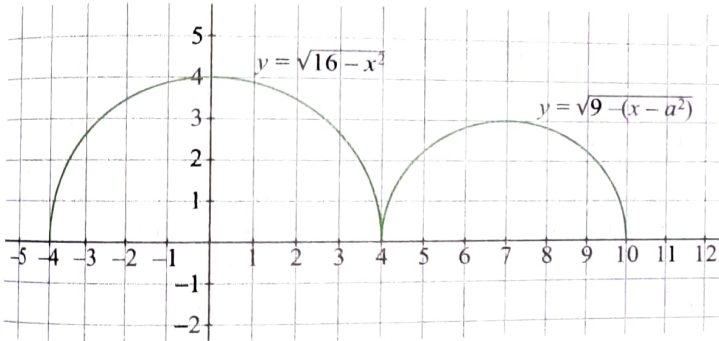
Hence, the equation of circle is $x^2 + y^2 = ab$.

70. (2) $y = \sqrt{9 - a^2 + 2ax - x^2}$

$$y = \sqrt{16 - x^2}$$

$$\Rightarrow y^2 + (x - a)^2 = 9$$

$$\Rightarrow y^2 + x^2 = 16$$



If $a > 7$, no solution exist. So, $a \leq 7$.

Hence, maximum value of a is 7.

71. (2) Let the coordinates of A, B, and C be (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) , respectively. Then, the chords of contact of tangents drawn from A, B, and C are $xx_1 + yy_1 = a^2$, $xx_2 + yy_2 = a^2$, and $xx_3 + yy_3 = a^2$, respectively. These three lines will be concurrent, if

$$\begin{vmatrix} x_1 & y_1 & -a^2 \\ x_2 & y_2 & -a^2 \\ x_3 & y_3 & -a^2 \end{vmatrix} = 0$$

or $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

or $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Therefore, points (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) are collinear.

72. (3) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points and $x^2 + y^2 = a^2$ be the given circle. Then, the chord of contact of tangents drawn from P to the given circle is $xx_1 + yy_1 = a^2$.

It will pass through $Q(x_2, y_2)$ if

$$x_1x_2 + y_1y_2 = a^2 \quad (1)$$

Now, $l_1 = \sqrt{x_1^2 + y_1^2 - a^2}$

$$l_2 = \sqrt{x_2^2 + y_2^2 - a^2}$$

and $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
 $= \sqrt{(x_2^2 + y_2^2) + (x_1^2 + y_1^2) - 2(x_1x_2 + y_1y_2)}$
 $= \sqrt{[(x_2^2 + y_2^2) + (x_1^2 + y_1^2) - 2a^2]} \quad [\text{Using Eq. (1)}]$
 $= \sqrt{(x_1^2 + y_1^2 - a^2) + (x_2^2 + y_2^2 - a^2)}$
 $= \sqrt{l_1^2 + l_2^2}$

73. (3) The equation of the line $y = x$ in distance form is

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta} = r, \text{ where } \theta = \frac{\pi}{4}$$

For point P, $r = 6\sqrt{2}$. Therefore, the coordinates of P are given by

$$\frac{x}{\cos(\pi/4)} = \frac{y}{\sin(\pi/4)} = 6\sqrt{2} \text{ or } x = 6, y = 6$$

Since P(6, 6) lies on $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$72 + 12(g + f) + c = 0 \quad (1)$$

Since $y = x$ touches the circle, the equation $2x^2 + 2x(g + f) + c = 0$ has equal roots. Therefore,

$$4(g + f)^2 = 8c$$

or $(g + f)^2 = 2c \quad (2)$

From (1), we get

$$[12(g + f)]^2 = [-(c + 72)]^2$$

or $144(g + f)^2 = (c + 72)^2$

or $144(2c) = (c + 72)^2$

or $(c - 72)^2 = 0$ or $c = 72$

74. (1) Any point on the line $x + y = 25$ is $P \equiv (a, 25 - a)$, $a \in R$.

The equation of chord AB is

$$T = 0$$

i.e., $xa + y(25 - a) = 9 \quad (1)$

If the midpoint of chord AB is $C(h, k)$, then the equation of chord AB is

$$T = S_1$$

i.e., $xh + yk = h^2 + k^2 \quad (2)$

Comparing the ratio of coefficients of (1) and (2), we get

$$\frac{a}{h} = \frac{25 - a}{k} = \frac{9}{h^2 + k^2}$$

or $\frac{a + 25 - a}{h + k} = \frac{9}{h^2 + k^2}$

Thus, the locus of C is $25(x^2 + y^2) = 9(x + y)$

75. (1) Let the center be $(0, \alpha)$. Then the equation of the circle is $x^2 + (y - \alpha)^2 = |a|^2$.

So, the equation of chord of contact for $P(h, k)$ is

$$xh + yk - \alpha(y + k) + \alpha^2 - a^2 = 0$$

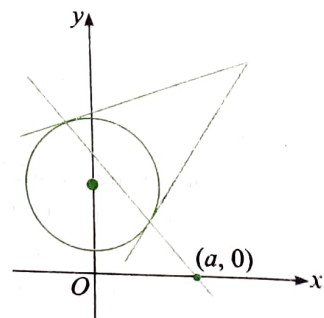
It passes through $(a, 0)$.

Therefore,

$$\alpha^2 - \alpha k + ah - a^2 = 0$$

As α is real,

$$k^2 - 4(ah - a^2) \geq 0$$



76. (3) Substituting $y = mx$ in the equation of circle, we get

$$x^2 + m^2x^2 + ax + bmx + c = 0$$

(y/x denotes the slope of the tangent from the origin on the circle.)

Since the line is touching the circle, we must have discriminant

$$(a + bm)^2 - 4c(1 + m^2) = 0$$

$$\text{or } a^2 + b^2m^2 + 2abm - 4c - 4cm^2 = 0$$

$$\text{or } m^2(b^2 - 4c) + 2abm + a^2 - 4c = 0$$

This equation has two roots m_1 and m_2 . Therefore,

$$m_1 + m_2 = -\frac{2ab}{b^2 - 4c} = \frac{2ab}{4c - b^2}$$

77. (2) The equation of the pair of tangents from $(0, 0)$ is

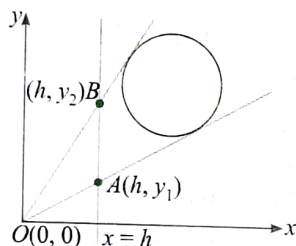
$$SS' = T^2$$

$$\text{or } (x^2 + y^2 + 2gx + 2fy + c) = (gx + fy + c)^2 \quad (1)$$

The intersection points of the above pair with $x = h$ are given by

$$(gh + fy + c)^2 = (h^2 + y^2 + 2gh + 2fy + c)c$$

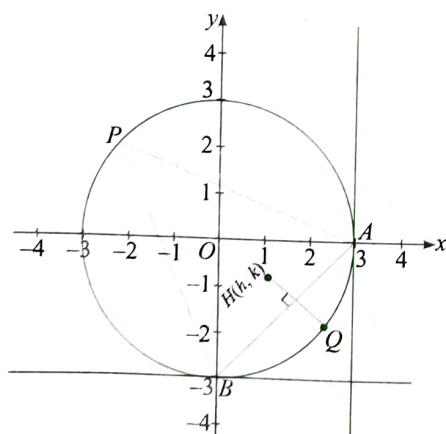
$$\text{or } (f^2 - c)y^2 + 2fghy + h^2(g^2 - c) = 0$$



If its roots are y_1 and y_2 , then the length of intercept

$$\begin{aligned} AB^2 &= |y_1 - y_2|^2 \\ &= (y_1 + y_2)^2 - 4y_1y_2 \\ &= \left(\frac{2fgh}{f^2 - c} \right)^2 - 4 \frac{h^2(g^2 - c)}{f^2 - c} \\ &= \frac{4ch^2}{(f^2 - c)^2} (g^2 + f^2 - c) \end{aligned}$$

78. (1)



The equation of AB is $x - y = 3$ (chord of contact)

Let orthocentre be $H(h, k)$.

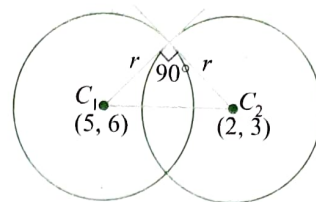
Its image w.r.t. AB is $Q(k+3, h-3)$.

Q will lie on the circle $x^2 + y^2 = 9$.

$$\therefore (k+3)^2 + (h-3)^2 = 9$$

So, required locus is $(y+3)^2 + (x-3)^2 = 9$.

79. (2)



$$2r^2 = 3^2 + 3^2 = 18 \text{ or } r^2 = 9 \text{ or } r = 3$$

80. (1) The center of the circle $x^2 + y^2 = 2x$ is $(1, 0)$. The common chord of the other two circles is

$$8x - 15y + 26 = 0$$

The distance from $(1, 0)$ to $8x - 15y + 26 = 0$ is

$$\frac{|8 + 26|}{\sqrt{15^2 + 8^2}} = 2$$

81. (2) Let the two circles be as shown in the figure.

The smaller circle is $x^2 + y^2 = 1$.

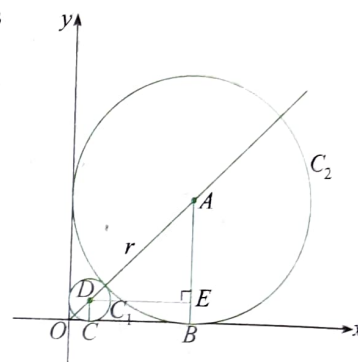
Clearly, from the figure

$$\cos 45^\circ = \frac{DE}{AD}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{r-1}{r+1}$$

$$\Rightarrow r+1 = \sqrt{2}r - \sqrt{2}$$

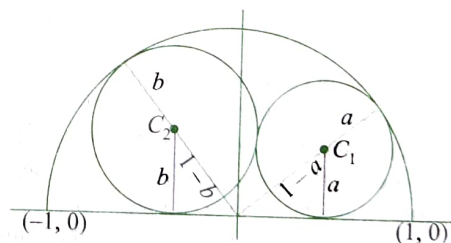
$$\Rightarrow r = \frac{\sqrt{2}+1}{\sqrt{2}-1} = 3 + 2\sqrt{2}$$



82. (1) a, b , and c are in AP. So, $ax + by + c = 0$ represents a family of lines passing through the point $(1, -2)$. So, the family of circles (concentric) will be given by $x^2 + y^2 - 2x + 4y + c = 0$. It intersects the given circle orthogonally. Therefore,

$$2(-1)(-2) + 2(2)(-2) = -1 + c \text{ or } c = -3$$

83. (1)



Let the centers of the circles be C_1 and C_2 .

Then C_1 is $(\sqrt{1-2a}, a)$ and C_2 is $(\sqrt{1-2b}, b)$. Now,

$$C_1C_2 = a + b = a + \frac{1}{2}$$

$$\text{or } 1 - 2a + \left(a - \frac{1}{2}\right)^2 = \left(a + \frac{1}{2}\right)^2$$

$$\text{or } a = \frac{1}{4}$$

84. (2) The common chord is $S_1 - S_2 = 0$.

$$4x - \mu y + 1 = 0$$

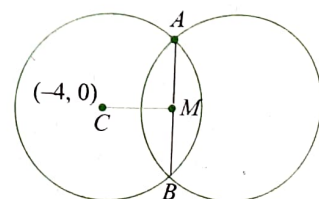
$$\therefore AC = \text{Radius of first circle} = \sqrt{15}$$

$$AM = \frac{AB}{2} = \sqrt{6}$$

$$\therefore CM = 3$$

$$\therefore \frac{|-16+1|}{\sqrt{16+\mu^2}} = 3$$

$$\text{or } \mu = \pm 3$$



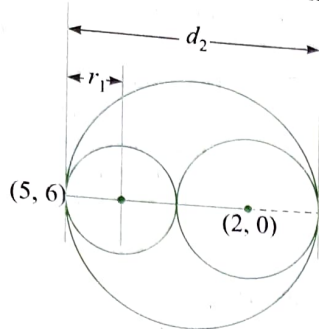
85. (2) The given circle is $(x-2)^2 + y^2 = 4$, the center is $(2, 0)$, and the radius is 2.
Therefore, distance between $(2, 0)$ and $(5, 6)$ is $\sqrt{9+36}$

$= 3\sqrt{5}$. Then

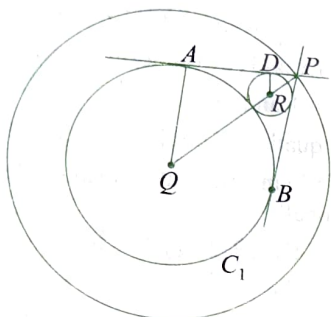
$$r_1 = \frac{3\sqrt{5}-2}{2}$$

$$\text{and } r_2 = \frac{d_2}{2} = \frac{3\sqrt{5}+2}{2}$$

$$\therefore r_1 r_2 = \frac{41}{4}$$



86. (4) $AQ = 3 + 2\sqrt{2}$
 $PQ = 3\sqrt{2} + 4$



Let r be the required radius. Then,

$$3\sqrt{2} + 4 = 3 + 2\sqrt{2} + r + r\sqrt{2}$$

$$\sqrt{2} + 1 = r(1 + \sqrt{2}) \text{ or } r = 1$$

87. (3) Let the equation of the two circles be

$$(x-r)^2 + (y-r)^2 = r^2$$

i.e., $x^2 + y^2 - 2rx - 2ry + r^2 = 0$, where $r = r_1, r_2$.

The condition of orthogonality gives

$$2r_1 r_2 + 2r_1 r_2 = r_1^2 + r_2^2 \text{ or } 4r_1 r_2 = r_1^2 + r_2^2 \quad (1)$$

Circle passes through (a, b) . Therefore,

$$a^2 + b^2 - 2ra - 2rb + r^2 = 0$$

i.e., $r^2 - 2r(a+b) + a^2 + b^2 = 0$

$$\therefore r_1 + r_2 = 2(a+b) \text{ and } r_1 r_2 = a^2 + b^2$$

$$\therefore 4(a^2 + b^2) = 4(a+b)^2 - 2(a^2 + b^2)$$

[From (1)]

$$\text{i.e., } a^2 - 4ab + b^2 = 0$$

88. (3) $C_1 \equiv (-1, -4); C_2 \equiv (2, 5)$

$$r_1 = \sqrt{1+16+23} = 2\sqrt{10}$$

$$r_2 = \sqrt{4+25-19} = \sqrt{10}$$

$$C_1 C_2 = \sqrt{9+81} = 3\sqrt{10}$$

$$\therefore C_1 C_2 = r_1 + r_2$$

Hence, the circles touch externally.

89. (3) The locus of the center of the circle cutting $S_1 = 0$ and $S_2 = 0$ orthogonally is the radical axis between $S_1 = 0$ and $S_2 = 0$, i.e.,

$$S_1 - S_2 = 0 \text{ or } 9x - 10y + 11 = 0.$$

90. (1) Equation of common chord of the circles is

$$-1 + (\lambda + 6)x - (8 - 2\lambda)y + 3 = 0 \quad (2)$$

$$\text{or } (\lambda + 6)x - (8 - 2\lambda)y + 2 = 0$$

Let the tangents at the extremities of common chord intersect at $P(h, k)$.

Thus, equation of chord of contact of circle $x^2 + y^2 = 1$ w.r.t. point P is

$$hx + ky = 1$$

Equations (1) and (2) represent the same straight line.

$$\therefore \frac{h}{\lambda + 6} = \frac{k}{2\lambda - 8} = \frac{-1}{2}$$

Eliminating λ , we get

$$2h - k + 10 = 0$$

$$\text{or } 2x - y + 10 = 0$$

91. (4) $C_1 \equiv (0, 0); C_2 \equiv (3\sqrt{3}, 3)$

$$\text{and } r_1 = 2, r_2 = 4$$

$$\therefore C_1 C_2 = r_1 + r_2$$

Thus, circles touch each other externally.

The equation of common tangent at P is

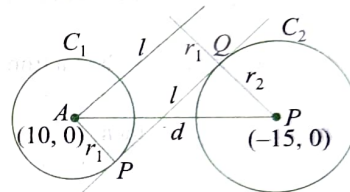
$$\sqrt{3}x + y - 4 = 0 \quad (1)$$

Comparing it with $x \cos \theta + y \sin \theta = 2$, we get $\theta = \pi/6$.

92. (1) The centers are $(10, 0)$ and $(-15, 0)$

and the radii are $r_1 = 6$ and $r_2 = 9$.

Also, $d = 25, r_1 + r_2 < d$.



So, the circles are neither intersecting nor touching. Therefore,

$$PQ = \sqrt{d^2 - (r_1 + r_2)^2} \\ = \sqrt{625 - 225} \\ = 20$$

93. (1) Let $\angle AB_1 B_2 = \theta$. Then,

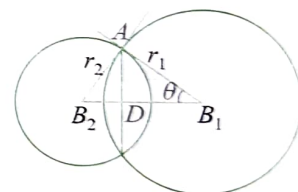
$$AD = r_1 \sin \theta \text{ (In } \triangle ADB_1)$$

$$\text{and } AD = r_2 \cos \theta \text{ (In } \triangle ADB_2)$$

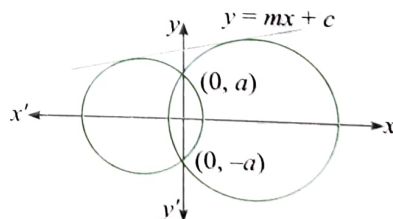
$$\therefore AD^2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) = 1$$

$$\text{or } AD = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

$$\text{Thus, length of common chord} = \frac{2r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$



94. (3)



The equation of the family of circles through $(0, a)$ and $(0, -a)$ is

$$[x^2 + (y-a)(y+a)] + \lambda x = 0, \lambda \in \mathbb{R}$$

$$\text{or } x^2 + y^2 + \lambda x - a^2 = 0$$

Since circles touch line $y = mx + c$,

$$\sqrt{\left(\frac{\lambda}{2}\right)^2 + a^2} = \frac{|(-m\lambda/2) + c|}{\sqrt{1+m^2}}$$

$$\text{or } (1+m^2)\left[\frac{\lambda^2}{4} + a^2\right] = \left(\frac{m\lambda}{2} - c\right)^2$$

$$\text{or } (1+m^2)\left[\frac{\lambda^2}{4} + a^2\right] = \frac{m^2\lambda^2}{4} - mc\lambda + c^2$$

$$\text{or } \lambda^2 + 4mc\lambda + 4a^2(1+m^2) - 4c^2 = 0$$

$$\therefore \lambda_1 \lambda_2 = 4[a^2(1+m^2) - c^2]$$

$$\text{or } g_1 g_2 = [a^2(1+m^2) - c^2]$$

$$\text{Now, } g_1 g_2 + f_1 f_2 = \frac{c_1 + c_2}{2} \quad (\text{As circles cut orthogonally.})$$

$$\therefore a^2(1+m^2) - c^2 = -a^2$$

$$\text{Hence, } c^2 = a^2(2+m^2)$$

95. (4) Let the center of the variable circle be (h, k) .

Since the circle touches the axis of y , its radius is $|h|$.

The radius of the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ is 2 and it has its center at $(3, 3)$.

Since the two circles touch each other externally,

Distance between the centers = Sum of the radii

$$\text{or } \sqrt{(h-3)^2 + (k-3)^2} = |h| + 2$$

$$\text{or } k^2 - 10h - 6k + 14 = 0 \quad (\text{Considering } h > 0)$$

Hence, the locus of (h, k) is $y^2 - 10x - 6y + 14 = 0$.

96. (1) Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points and $x^2 + y^2 = a^2$ be the circle.

The chord of contact of tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.

If it passes through $Q(x_2, y_2)$, then

$$x_1 x_2 + y_1 y_2 = a^2 \quad (1)$$

The equation of the circle on PQ as diameter is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

$$\text{or } x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1 x_2 + y_1 y_2 = 0$$

This circle will cut the given circle orthogonally if

$$0 + 0 = -a^2 + x_1 x_2 + y_1 y_2$$

$$\text{or } x_1 x_2 + y_1 y_2 - a^2 = 0$$

which is true by (1).

97. (2) Let A and B be the centers and r_1 and r_2 be the radii of the two circles. Then,

$$A \equiv \left(-\frac{1}{2}, -\frac{1}{2}\right), B \equiv \left(-\frac{1}{2}, \frac{1}{2}\right),$$

$$r_1 = \frac{1}{\sqrt{2}}, r_2 = \frac{1}{\sqrt{2}}$$

$$\cos \theta = \frac{r_1^2 + r_2^2 - AB^2}{2r_1 r_2}$$

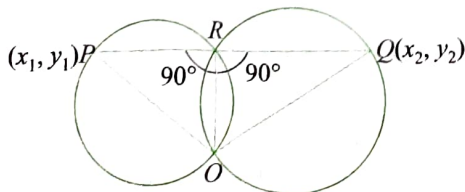
$$= \frac{\frac{1}{2} + \frac{1}{2} - 1}{2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = 0$$

$$\text{or } \theta = \frac{\pi}{2}$$

Therefore, the required line is parallel to the x -axis and since it passes through $(1, 2)$, its equation will be $y = 2$.

98. (2) $OR = \frac{2 \times \text{area of } \triangle OPQ}{PQ}$

$$= \frac{2 \times \left| \frac{1}{2} (x_1 y_2 - x_2 y_1) \right|}{PQ} = \frac{|x_1 y_2 - x_2 y_1|}{PQ}$$



99. (3) The equation of radical axis (i.e., common chord) of the two circles is

$$10x + 4y - a - b = 0 \quad (1)$$

The center of the first circle is $H(-4, -4)$.

Since the second circle bisects the circumference of the first circle, the center $H(-4, -4)$ of the first circle must lie on the common chord (1). Therefore,

$$-40 - 16 - a - b = 0$$

$$\text{or } a + b = -56$$

100. (1) $xy - 2x - y + 2 = 0$

$$\Rightarrow (x-1)(y-2) = 0$$

$$\Rightarrow x = 1 \text{ and } y = 2$$

Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

So, centre is $(1, 2)$ (as normal intersects at centre of the circle).

i.e., $-g = 1$ and $-f = 2$

Thus, equation of circle is $x^2 + y^2 - 2x - 4y + c = 0$.

This circle intersects the given circle $x^2 + y^2 + 2x + 4y - 4 = 0$ orthogonally.

$$\therefore 2(-1)(1) + 2(-2)(2) = c - 4$$

$$\Rightarrow c = -6$$

Hence, required circle is $x^2 + y^2 - 2x - 4y - 6 = 0$.

101. (3) $S_1 \equiv x^2 + y^2 - 4y - 5 = 0$

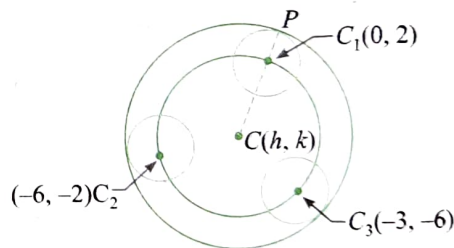
$$S_2 \equiv x^2 + y^2 + 12x + 4y + 31 = 0$$

and $S_3 \equiv x^2 + y^2 + 6x + 12y + 36 = 0$

$$C_1 \equiv (0, 2), r_1 = 3$$

$$C_2 \equiv (-6, -2), r_2 = 3$$

$$C_3 \equiv (-3, -6), r_3 = 3$$



All circles have the same radius.

The radius of the circle touching all the circles is

$$CP = CC_1 + C_1P = CC_1 + 3$$

Let $C(h, k)$ be the center of the required circle. Then,

$$CC_1 = CC_2 = CC_3$$

$$\text{or } CC_1^2 = CC_2^2 = CC_3^2$$

$$\text{or } (h-0)^2 + (k-2)^2 = (h+6)^2 + (k+2)^2$$

$$= (h+3)^2 + (k+6)^2$$

$$\text{or } -4k + 4 = 12h + 4k + 40 = 6h + 12k + 45$$

$$\text{or } 3h + 2k + 9 = 0 \text{ and } 6h - 8k - 5 = 0$$

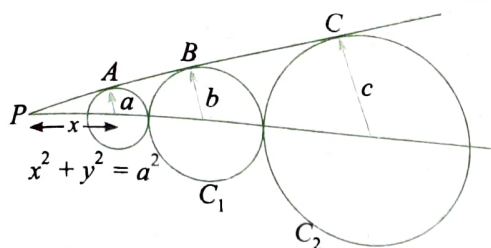
Solving, we get

$$h = \frac{-31}{18}, k = \frac{-23}{12}$$

$$\therefore CC_1 = \sqrt{\left(0 + \frac{31}{18}\right)^2 + \left(2 + \frac{23}{12}\right)^2} = \frac{5}{36} \sqrt{949}$$

$$\text{Now, } CP = CC_1 + 3 = \frac{5}{36} \sqrt{949} + 3$$

102. (2)



From the figure,

$$\frac{a}{x} = \frac{b}{x+a+b} = \frac{c}{x+a+2b+c}$$

$$\therefore ax + a^2 + ab = bx \quad (\text{Comparing first two})$$

$$\text{or } x = \frac{a^2 + ab}{b-a}$$

$$ax + a^2 + 2ab + ac = cx \quad (\text{Comparing first and third})$$

$$\text{or } x = \frac{a^2 + 2ab + ac}{c-a}$$

$$\therefore \frac{a^2 + ab}{b-a} = \frac{a^2 + 2ab + ac}{c-a}$$

$$a^2c - a^3 + abc - a^2b = a^2b + 2ab^2 + abc - a^3 - 2a^2b - a^2c$$

$$\text{or } ac = b^2$$

Therefore, a , b , and c are in GP.

103. (1) Let the equation of the circle through (a, b) be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

$$\text{So, } a^2 + b^2 + 2ag + 2bf + c = 0 \quad (2)$$

Since circle (1) cuts $x^2 + y^2 = k^2$ orthogonally, we have

$$2g(0) + 2f(0) = c - k^2 \text{ or } c = k^2$$

Putting $c = k^2$ in (2), we get

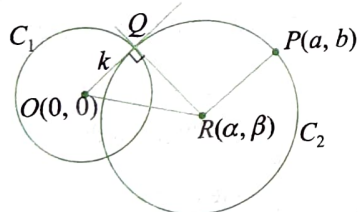
$$2ag + 2bf + (a^2 + b^2 + k^2) = 0$$

So, the locus of the center $(-g, -f)$ is

$$-2ax - 2by + (a^2 + b^2 + k^2) = 0$$

$$\text{or } 2ax + 2by - (a^2 + b^2 + k^2) = 0$$

Alternative Method:



As shown in the figure, circle C_2 intersect the circle $C_1: x^2 + y^2 = k^2$ orthogonally at point Q .

Circle C_2 passes through the point $P(a, b)$. We have to find the locus of center $R(\alpha, \beta)$ of circle C_2 .

Now from the figure, $RQ = RP = \text{radius of the circle } C_2$

$$\therefore RP^2 = RQ^2$$

$$\therefore RP^2 = OR^2 - OQ^2$$

$$\therefore (\alpha - a)^2 + (\beta - b)^2 = (\alpha^2 + \beta^2) - k^2$$

$$\therefore 2a\alpha + 2b\beta - (a^2 + b^2 + k^2) = 0$$

$$\therefore \text{locus is } 2ax + 2by - (a^2 + b^2 + k^2) = 0$$

104. (3) Circles

$$S_1: x^2 + y^2 - 2y - 3 = 0 \text{ and}$$

$$S_2: x^2 + y^2 - 8x - 18y + 93 = 0$$

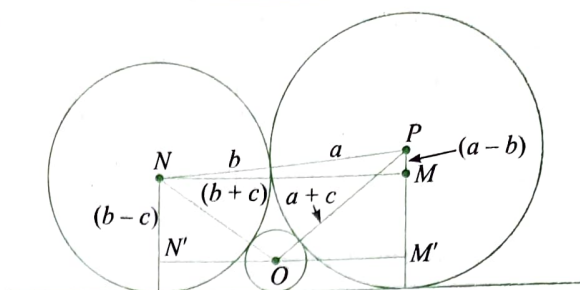
$$C_1 \equiv (0, 1) \text{ and } r_1 = 2$$

$$C_2 \equiv (4, 9) \text{ and } r_2 = 2$$

Since circles have same radius, centre C_3 of the smallest circle is collinear with C_1 and C_2 and is midpoint of C_1C_2 .

$$\therefore C_3 \equiv (2, 5)$$

105. (3)



$$\therefore MN' = MN$$

$$\text{Now, } MN = \sqrt{(a+b)^2 - (a-b)^2} = \sqrt{4ab}$$

$$N'O = \sqrt{(b+c)^2 - (b-c)^2} = \sqrt{4bc}$$

$$M'O = \sqrt{(a+c)^2 - (a-c)^2} = \sqrt{4ac}$$

$$\text{Now, } N'O + M'O = MN$$

$$\Rightarrow 2\sqrt{ab} = 2\sqrt{bc} + 2\sqrt{ac}$$

Dividing by $\frac{1}{\sqrt{abc}}$, we get

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

$$106. (3) \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \tan \theta = \frac{2}{3}$$

$$\therefore \sin \theta = \frac{2}{\sqrt{13}} \text{ and } \cos \theta = \frac{3}{\sqrt{13}}$$

$$\therefore A' \equiv (OA \cos \theta, OA \sin \theta)$$

$$\therefore A' \equiv (3, 2)$$

$$\text{Similarly, } B' \equiv (OB \cos \theta, OB \sin \theta) \equiv (6, 4)$$

$$\text{Now, } A'B' = r_1 + r_2$$

So, circles touch each other externally.

The point of contact C divides C_1C_2 internally in ratio $r_1 : r_2$.

$$\therefore C \equiv \left(\frac{3 \cdot \frac{\sqrt{13}}{3} + 6 \cdot \frac{2\sqrt{13}}{3}}{\frac{\sqrt{13}}{3} + \frac{2\sqrt{13}}{3}}, \frac{2 \cdot \frac{\sqrt{13}}{3} + 4 \cdot \frac{2\sqrt{13}}{3}}{\frac{\sqrt{13}}{3} + \frac{2\sqrt{13}}{3}} \right) \equiv \left(5, \frac{10}{3} \right)$$

Therefore, required radical axis is a line perpendicular to $A'B'$ and is passing through point C , which is

$$y - \frac{10}{3} = -\frac{3}{2}(x - 5)$$

$$107. (3) \text{ Let the second circle be } x^2 + y^2 + 2gx + 2fy = 0.$$

But $y = x$ touches the circle,

Hence, $x^2 + x^2 + 2gx + 2fx = 0$ has equal roots, i.e., $f + g = 0$.

Therefore, the equation of the common chord is $2(g-3)x + 2(-g-4)y + 7 = 0$

$$\text{or } (-6x - 8y + 7) + g(2x - 2y) = 0$$

which passes through the point of intersection of $-6x - 8y + 7 = 0$ and $2x - 2y = 0$, i.e., $(1/2, 1/2)$.

$$108. (4) \text{ Let the equation of the circle be}$$

$$x^2 + y^2 - 4 + k(2x + y - 1) = 0$$

where k is a real number.

$$\text{Radius} = \sqrt{\frac{5k^2}{4} + 4 + k}$$

Radius is minimum when $k = -2/5$.

Therefore, the required equation will be

$$5x^2 + 5y^2 - 4x - 2y - 18 = 0$$

109. (4) The equation of any circle through the points of intersection of the given circles is
- $$x^2 + y^2 - 4x - 2y - 8 + k(x^2 + y^2 - 2x - 4y - 8) = 0 \quad (1)$$
- Since circle (1) passes through $(-1, 4)$, we have
- $$k = 1$$
- Therefore, the required circle is
- $$x^2 + y^2 - 3x - 3y - 8 = 0$$

Multiple Correct Answers Type

1. (1), (3)

The equation of radical axis of the given circle is $x = 0$.
If one circle lies completely inside the other, the centers of both circles should lie on the same side of the radical axis and the radical axis should not intersect the circles. Therefore,

$$(-a_1)(-a_2) > 0$$

$$\text{or } a_1 a_2 > 0$$

Also, $y^2 + c = 0$ should have imaginary roots, i.e., $c > 0$.

2. (1), (3), (4)

The coordinates of O are $(5, 3)$ and the radius is 2.

The equation of tangent at $A(7, 3)$ is

$$7x + 3y - 5(x + 7) - 3(y + 3) + 30 = 0$$

$$\text{i.e., } 2x - 14 = 0$$

$$\text{i.e., } x = 7$$

The equation of tangent at $B(5, 1)$ is

$$5x + y - 5(x + 5) - 3(y + 1) + 30 = 0$$

$$\text{i.e., } -2y + 2 = 0$$

$$\text{i.e., } y = 1$$

Therefore, the coordinates of C are $(7, 1)$. So, area of $OACB = 4$

The equation of AB is $x - y = 4$ (radical axis).

The equation of the smallest circle is

$$(x - 7)(x - 5) + (y - 3)(y - 1) = 0$$

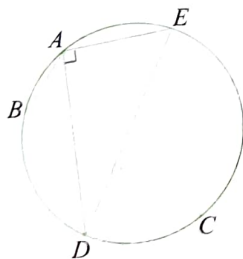
$$\text{i.e., } x^2 + y^2 - 12x - 4y + 38 = 0$$

3. (2), (3)

The equation of director circle is $x^2 + y^2 = 25$. The point $(\alpha, 3)$ must lie on this circle. So,

$$\alpha^2 + 9 = 25 \text{ or } \alpha^2 = 16 \text{ or } \alpha = \pm 4$$

4. (1), (2), (3)



Orthocentre of $\triangle ADE$ is A itself and its circumcentre is at origin. Hence, centroid is also a fixed point.

5. (1), (3)

The equation of any line through the origin $(0, 0)$ is

$$y = mx$$

If line (i) is tangent to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, then the length of perpendicular from center (r, h) on (i) is equal to the radius of the circle, i.e.,

$$\frac{|mr - h|}{\sqrt{m^2 + 1}} = \sqrt{r^2 + h^2 - h^2}$$

$$(mr - h)^2 = (m^2 + 1)r^2$$

$$0 \cdot m^2 + (2hr)m + (r^2 - h^2) = 0$$

$$m = \infty, \frac{h^2 - r^2}{2hr}$$

Substituting these values in (i), we get the tangents as

$$x = 0 \text{ and } (h^2 - r^2)x - 2rhy = 0$$

6. (1), (2), (3), (4)

Putting $y = c^2/x$ in $x^2 + y^2 = a^2$, we get

$$x^2 + \frac{c^4}{x^2} = a^2$$

$$\text{or } x^4 - a^2 x^2 + c^4 = 0$$

As x_1, x_2, x_3 , and x_4 are the roots of (i), we have

$$x_1 + x_2 + x_3 + x_4 = 0 \text{ and } x_1 x_2 x_3 x_4 = c^4$$

Similarly, forming equation in y , we get

$$y_1 + y_2 + y_3 + y_4 = 0 \text{ and } y_1 y_2 y_3 y_4 = c^4$$

7. (1), (3), (4)

$$x^2 + y^2 + 8x - 10y - 40 = 0$$

The center of the circle is $(-4, 5)$.

its radius is 9.

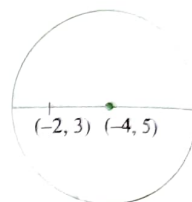
distance of the center $(-4, 5)$ from the point $(-2, 3)$ is $\sqrt{4 + 4} = 2\sqrt{2}$. Therefore,

$$a = 2\sqrt{2} + 9 \text{ and } b = -2\sqrt{2} + 9$$

$$\therefore a + b = 18$$

$$a - b = 4\sqrt{2}$$

$$a \cdot b = 81 - 8 = 73$$



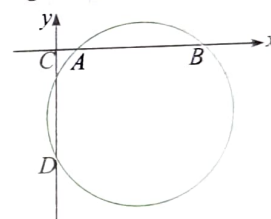
8. (1), (2), (3), (4)

The given circle is $x^2 + y^2 + 2hxy + 2gx + 2fy + c = 0$

For (i) to represent a circle, $h = 0$.

So, the given circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$



For circle (ii) to pass through three quadrants only,

I. Circle must cut x -axis. i.e., $g^2 - c > 0$.

II. Circle must cut y -axis. $f^2 - c > 0$.

III. The origin should lie outside circle (ii), i.e., $c > 0$.

Therefore, the required conditions are $g^2 > c, f^2 > c, c > 0$, and $h = 0$.

9. (1), (3)

The point from which the tangents drawn are at right angle lies on the director circle.

Equation of director circle is $x^2 + y^2 = 2 \times 16 = 32$.

Putting $x = 2$, we get

$$y^2 = 28$$

$$\text{or } y = \pm 2\sqrt{7}$$

So, the points can be $(2, 2\sqrt{7})$ or $(2, -2\sqrt{7})$.

10. (2), (4)

Line pair is $(x - 1)^2 - y^2 = 0$, i.e., $x + y - 1 = 0$ and $x - y - 1 = 0$.

Let the center be $(\alpha, 0)$. Then its distance from $x + y - 1 = 0$ is

$$\left| \frac{\alpha - 1}{\sqrt{2}} \right| = 2 \text{ (radius)}$$

$$\text{i.e., } \alpha = 1 \pm 2\sqrt{2}$$

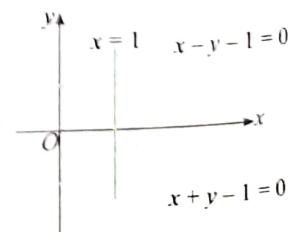
So, the center may be $(1 + 2\sqrt{2}, 0)$, $(1 - 2\sqrt{2}, 0)$.

Now, let the center be $(1, \beta)$. Then,

$$\left| \frac{1 + \beta - 1}{\sqrt{2}} \right| = 2$$

$$\text{or } \beta = \pm 2\sqrt{2}$$

So, the center may be $(1, 2\sqrt{2})$, $(1, -2\sqrt{2})$.



11. (1), (4)

The equation of the radical axis is

$$2\alpha x + 2y + 10 = 0$$

$$\text{i.e., } \alpha x + y + 5 = 0$$

Putting the value of y from (i) in the circle $x^2 + y^2 = 9$, we get

$$(1 + \alpha^2)x^2 + 10\alpha x + 16 = 0$$

Radical axis is tangent. So,

$$D = 0$$

$$\text{or } 36\alpha^2 - 64 = 0$$

$$\text{or } \alpha = \pm \frac{4}{3}$$

12. (2), (3)

$$x^2 + y^2 - 8x - 16y + 60 = 0$$

Equation of chord of contact from $(-2, 0)$ is

$$-2x - 4(x - 2) - 8y + 60 = 0$$

$$\text{or } 3x + 4y - 34 = 0$$

Solving (i) and (ii), we get

$$x^2 + \left(\frac{34 - 3x}{4}\right)^2 - 8x - 16\left(\frac{34 - 3x}{4}\right) + 60 = 0$$

$$\text{or } 16x^2 + 1156 - 204x + 9x^2 - 128x - 2176 + 192x + 960 = 0$$

$$\text{or } 5x^2 - 28x - 12 = 0$$

$$\text{or } (x - 6)(5x + 2) = 0$$

$$\text{or } x = 6, -\frac{2}{5}$$

Therefore, the points are $(6, 4)$ and $(-2/5, 44/5)$.

13. (1), (3)

The equation of any tangent to the circle $x^2 + y^2 = 25$ is of the form

$$y = mx + 5\sqrt{1 + m^2} \quad (\text{where } m \text{ is the slope})$$

It passes through $(-2, 11)$. Therefore,

$$11 = -2m + 5\sqrt{1 + m^2}$$

$$\text{or } (11 + 2m)^2 = 25(1 + m^2)$$

$$\text{or } m = \frac{24}{7}, -\frac{4}{3}$$

Therefore, the equation of tangent is

$$24x - 7y + 125 = 0$$

$$\text{or } 4x + 3y = 25$$

14. (1), (4)

Area of quadrilateral $= \sqrt{c} \times \sqrt{9 + 25 - c} = 15$

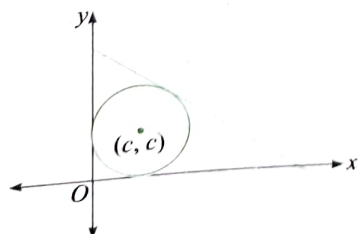
$$\therefore c = 9, 25$$

15. (1), (4)

We must have

$$\left| \frac{\frac{c}{3} + \frac{c}{4} - 1}{\sqrt{\frac{1}{3^2} + \frac{1}{4^2}}} \right| = c$$

$$\text{or } c = 6, 1$$



16. (1), (2), (3), (4)

Chords equidistant from the center are equal.

17. (2), (4)

Let the equation of the tangent be

$$x - 2y = k$$

Line (i) touches the circle. Therefore,

Distance from center to line (i) = Radius of circle

$$\text{or } \frac{|2 - 2 - k|}{\sqrt{5}} = \sqrt{4 + 1 + 15}$$

$$\text{or } |k| = 10 \text{ or } k = \pm 10$$

Therefore, the tangents can be $x - 2y \pm 10 = 0$.

18. (2), (3)

For the given circles $S_1: x^2 + y^2 - 2x - 4y + 1 = 0$ and

$$S_2: x^2 + y^2 + 4x + 4y - 1 = 0,$$

$$C_1 \equiv (1, 2), r_1 = 2, C_2 \equiv (-2, -2), \text{ and } r_2 = 3$$

Now, $r_1 + r_2 = 5$ and $C_1 C_2 = 5$.Hence, the circles touch externally. Also, the common tangent at the point of contact is $S_1 - S_2 = 0$ or $3x + 4y - 1 = 0$.

19. (1), (2), (3), (4)

$$r_1 = 5; r_2 = \sqrt{15}; C_1 C_2 = \sqrt{40}$$

$$\therefore r_1 + r_2 > C_1 C_2 > r_1 - r_2$$

Hence, the circles intersect at two distinct points.

There are two common tangents. Also,

$$2g_1 g_2 + 2f_1 f_2 = 2(1)(3) + 2(2)(-4) = -10$$

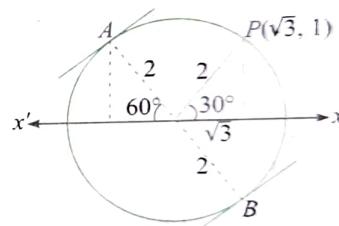
$$\text{and } c_1 + c_2 = -20 + 10 = -10$$

Thus, the two circles are orthogonal.

$$\text{Length of common chord} = \frac{2r_1 r_2}{\sqrt{r_1^2 + r_2^2}} = 5\sqrt{\frac{3}{2}}$$

$$\text{Length of common tangent} = \sqrt{C_1 C_2^2 - (r_1 - r_2)^2} = 5 \left(\frac{12}{5} \right)^{1/4}$$

20. (2), (4)

Clearly, $A \equiv (-2 \cos 60^\circ, 2 \sin 60^\circ)$ and $B \equiv (2 \cos 60^\circ, -2 \sin 60^\circ)$.The tangent at A is $x(-2 \cos 60^\circ) + y(2 \sin 60^\circ) = 4$ and that at B is $x(2 \cos 60^\circ) + y(-2 \sin 60^\circ) = 4$.

21. (2), (3)

The given circles are $x^2 + y^2 - 2x = 0, x > 0$, and $x^2 + y^2 + 2x = 0, x < 0$.

From the figure, the centers of the

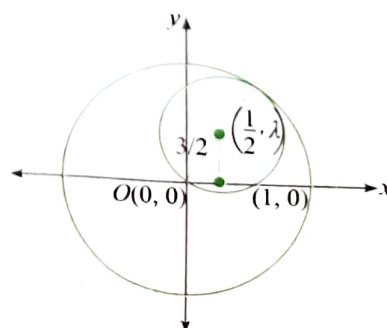
required circles will be $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

Therefore, the equations of the

circles are

$$(x - 0)^2 + (y \mp \sqrt{3})^2 = 1^2$$

22. (3), (4)

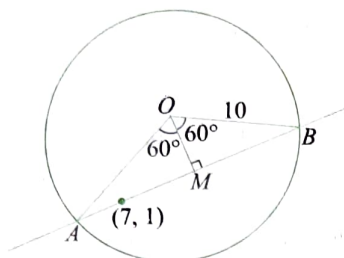


From the diagram,

$$\sqrt{\left(\frac{1}{2}\right)^2 + \lambda^2} = \frac{3}{2} \text{ or } \lambda = \pm\sqrt{2}$$

Hence, the centers of the circle are $(1/2, \pm\sqrt{2})$.

23. (1), (2)



Point of intersection of given lines is $(7, 1)$, which lies inside the circle $x^2 + y^2 = 100$.

If line chord AB through this point divides circumference of the circle into two arcs whose lengths are in the ratio $2 : 1$, then AB subtends an angle of 120° at the centre.

So, from the figure, $OM = 10 \cos 60^\circ = 5$

Let the equation of line through $(7, 1)$ be

$$y - 1 = m(x - 7)$$

$$\text{or } mx - y - 7m + 1 = 0$$

$$OM = 5$$

$$\Rightarrow \left| \frac{1 - 7m}{\sqrt{1 + m^2}} \right| = 5$$

$$\Rightarrow (1 - 7m)^2 = 25(1 + m^2)$$

$$\Rightarrow 24m^2 - 14m - 24 = 0$$

$$\Rightarrow (4m + 3)(6m - 8) = 0$$

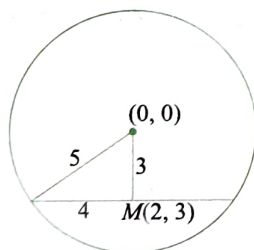
$$\Rightarrow m = -\frac{3}{4}, m = \frac{4}{3}$$

Putting the value of m in (i), we get

$$y - 1 = -\frac{3}{4}(x - 7) \text{ and } y - 1 = \frac{4}{3}(x - 7)$$

$$\text{or } 3x + 4y - 25 = 0 \text{ and } 4x - 3y - 25 = 0$$

24. (1), (2)



Let the slope of required line be m .

$$y - 3 = m(x - 2)$$

$$\Rightarrow mx - y + (3 - 2m) = 0$$

Length of perpendicular from origin = 3

$$\Rightarrow 9 + 4m^2 - 12m = 9 + 9m^2$$

$$\Rightarrow 5m^2 + 12m = 0$$

$$\Rightarrow m = 0, -\frac{12}{5}$$

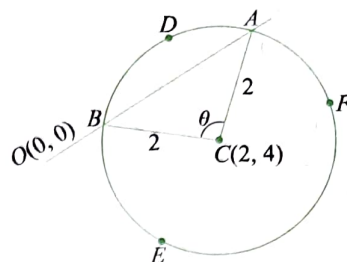
Hence, lines are

$$y - 3 = 0 \text{ or } y = 3$$

$$\text{and } y - 3 = -\frac{12}{5}(x - 2) \text{ or } 12x + 5y = 39$$

25. (1), (4)

Normal to the circle $x^2 + y^2 = 4$ passes through its centre $(0, 0)$.



So, equation of normal is $y = mx$.

Now, according to the question,

$$\frac{\text{Area of region } ADBA}{\text{Area of region } ABEFA} = \frac{\pi - 2}{3\pi + 2}$$

$$\therefore \frac{\left[\frac{\pi(2)^2}{2\pi} \cdot \theta - \frac{1}{2} (2)^2 \sin \theta \right]}{\pi(2)^2 - \left[\frac{\pi(2)^2}{2\pi} \cdot \theta - \frac{1}{2} (2)^2 \sin \theta \right]} = \frac{\pi - 2}{3\pi + 2}$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

$$\therefore \text{Distance of centre of circle from line} = 2 \cos \frac{\pi}{4}$$

$$\Rightarrow \frac{|4 - 2m|}{\sqrt{1 + m^2}} = \sqrt{2}$$

$$\Rightarrow m = 1, 7$$

Linked Comprehension Type

1. (4), 2. (2), 3. (1)

It is given that one of the diagonals of the square is parallel to the line $y = x$.

Also, the length of the diagonal of the square is $4\sqrt{2}$.

Hence, the equation of one of the diagonals is

$$\frac{x - 3}{1/\sqrt{2}} = \frac{y - 4}{1/\sqrt{2}} = r = \pm 2\sqrt{2}$$

Hence, $x - 3 = y - 4 = \pm 2$

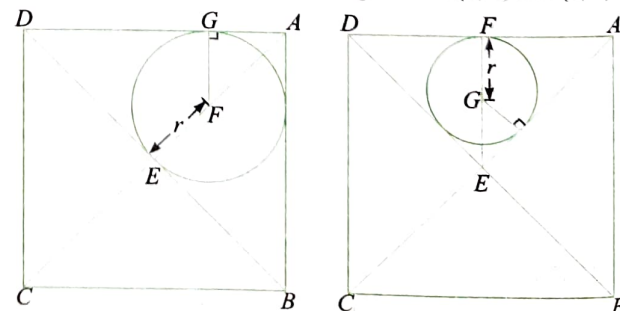
or $x = 5, 1$ and $y = 6, 2$

Hence, two of the vertices are $(1, 2)$ and $(5, 6)$.

The other diagonal is parallel to the line $y = -x$, so that its equation is

$$\frac{x - 3}{-1/\sqrt{2}} = \frac{y - 4}{1/\sqrt{2}} = r = \pm 2\sqrt{2}$$

Hence, the two vertices on this diagonal are $(1, 6)$ and $(5, 2)$.



(a)

(b)

$$AB = 4, AC = 4\sqrt{2}$$

$$\therefore AE = 2\sqrt{2}$$

In first figure, $EF + FA = AE$

or $r + \sqrt{2}r = 2\sqrt{2}$

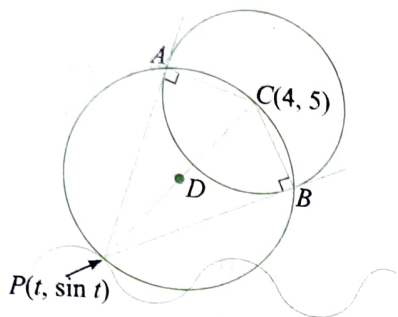
or $r = \frac{2\sqrt{2}}{\sqrt{2} + 1} = 2\sqrt{2}(\sqrt{2} - 1)$

In second figure, $EG + GF = EF$

or $\sqrt{2}r + r = 2$

or $r = \frac{2}{\sqrt{2} + 1} = 2(\sqrt{2} - 1)$

4. (4), 5. (3), 6. (3)



The center of the given circle is $C(4, 5)$. Points P, A, C , and B are concyclic such that PC is the diameter of the circle. Hence, the center $D(h, k)$ of the circumcircle of $\triangle ABC$ is the midpoint of PC . Then, we have

$$h = \frac{t+4}{2} \text{ and } k = \frac{\sin t + 5}{2}$$

Eliminating t , we have

$$k = \frac{\sin(2h-4) + 5}{2}$$

or $y = f(x) = \frac{\sin(2x-4) + 5}{2}$

$$\therefore f^{-1}(x) = \frac{\sin^{-1}(2x-5) + 4}{2}$$

Thus, the range of

$$y = \frac{\sin(2x-4) + 5}{2}$$

is $[2, 3]$ and the period is π .

Also, $f(x) = 4$, i.e., $\sin(2x-4) = 3$ which has no real solutions.

For $f(x) = 1$, $\sin(2x-4) = -3$ which has no real solutions.

But range of $y = \frac{\sin^{-1}(2x-5) + 4}{2}$ is $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$

7. (3), 8. (4), 9. (3)

The equation of the line passing through the points $A(3, 7)$ and $B(6, 5)$ is

$$y - 7 = -\frac{2}{3}(x - 3)$$

or $2x + 3y - 27 = 0$

Also, the equation of the circle with A and B as the endpoints of diameter is

$$(x-3)(x-6) + (y-7)(y-5) = 0$$

Now, the equation of the family of circles through A and B is

$$(x-3)(x-6) + (y-7)(y-5) + \lambda(2x+3y-27) = 0 \quad (1)$$

If the circle belonging to this family touches the x -axis, then equation $(x-3)(x-6) + (0-7)(0-5) + \lambda\{2x+3(0)-27\} = 0$ has two equal roots, for which discriminant $D = 0$. It gives two values of λ .

The equation of the common chord of (1) and $x^2 + y^2 - 4x - 6y - 3 = 0$ is the radical axis, which is

$$[(x-3)(x-6) + (y-7)(y-5) + \lambda(2x+3y-27)] - [x^2 + y^2 - 4x - 6y - 3] = 0$$

or $(2\lambda-5)x + (3\lambda-6)y + (-27\lambda+56) = 0$

or $(-5x-6y+56) + \lambda(2x+3y-27) = 0$

This is the family of lines which passes through the point of intersection of $-5x-6y+56=0$ and $2x+3y-27=0$, i.e., $(2, 23/3)$.

If circle (i) cuts $x^2 + y^2 = 29$ orthogonally, then

$$0 + 0 = -29 + 56 - 27\lambda = 0 \text{ or } \lambda = 1$$

So, the required circle is $x^2 + y^2 - 7x - 9y + 26 = 0$ and the center is $(7/2, 9/2)$.

10. (3), 11. (1), 12. (3)

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad (1)$$

The line $lx + my + 1 = 0$ will touch circle (1) if the length of perpendicular from the center $(-g, -f)$ of the circle on the line is equal to its radius, i.e.,

$$\frac{|-gl - mf + 1|}{\sqrt{l^2 + m^2}} = \sqrt{g^2 + f^2 - c}$$

$$(gl + mf - 1)^2 = (l^2 + m^2)(g^2 + f^2 - c)$$

or $(c - f^2)l^2 + (c - g^2)m^2 - 2gl - 2fm + 2gflm + 1 = 0 \quad (2)$

But the given condition is

$$4l^2 - 5m^2 + 6l + 1 = 0 \quad (3)$$

Comparing (ii) and (iii), we get

$$c - f^2 = 4, c - g^2 = -5, -2g = 6, -2f = 0, 2gf = 0$$

Solving, we get

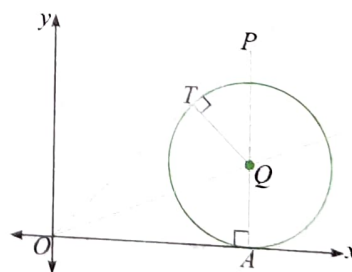
$$f = 0, g = -3, c = 4$$

Substituting these values in (1), the equation of the circle is $x^2 + y^2 - 6x + 4 = 0$. Any point on the line $x + y - 1 = 0$ is $(t, 1-t)$, $t \in R$.

The chord of contact w.r.t. this point of the circle is $tx + y(1-t) - 3(t+x) + 4 = 0$ or $t(x-y-3) + (-3x+y+4) = 0$, which is concurrent at the point of intersection of the lines $x - y - 3 = 0$ and $-3x + y + 4 = 0$ for all values of t . Hence, the lines are concurrent at $(1/2, -5/2)$.

Also, point $(2, -3)$ lies outside the circle from which two tangents can be drawn.

13. (3) Given $QT = QA = 1$.



Let $PQ = x$. Then $PT = \sqrt{x^2 - 1}$.

$\triangle TQP$ and $\triangle AOP$ are similar triangles. Then,

$$\frac{OA}{TQ} = \frac{AP}{TP}$$

$$\Rightarrow OT = OA = \frac{x+1}{\sqrt{x^2-1}}$$

Perimeter of $\triangle OAP$ is 8 units.

$$\therefore 1 + x + \frac{2(x+1)}{\sqrt{x^2-1}} + \sqrt{x^2-1} = 8 \text{ or } x = \frac{5}{3}$$

14. (1) Now, $OA = 2$.

So, centre of the circle is $Q(2, 1)$.

Equation of circle is $(x-2)^2 + (y-1)^2 = 1$.

15. (1) $AP = AQ + PQ = 1 + 5/3 = 8/3$

So, slope of OP is $4/3$.

Hence, equation of OP is $4x - 3y = 0$.

16. (2) $PQ = PR$, i.e., parallelogram $PQSR$ is a rhombus. Therefore,

Midpoint of $QR \equiv$ Midpoint of PS

and $QR \perp PS$

Therefore, S is the mirror image of P in QR .

$$L \equiv 2x + y = 6$$

Let $P \equiv (k, 6 - 2k)$.

$$\angle PQO = \angle PRO = \frac{\pi}{2}$$

Therefore, OP is the diameter of circumcircle PQR . Then the center is

$(k/2, 3 - k)$. So,

$$x = \frac{k}{2} \text{ or } k = 2x \text{ and } y = 3 - k$$

Therefore, the required locus is $2x + y = 3$.

17. (4) $P \equiv (6, 8)$

Therefore, the equation of QR (chord of contact) is

$$6x + 8y = 4$$

$$\text{or } 3x + 4y - 2 = 0$$

$$\therefore PM = \frac{48}{5} \text{ and } PQ = \sqrt{96}$$

$$QM = \sqrt{96 - \frac{48^2}{25}} = \sqrt{\frac{96}{25}}$$

$$\therefore QR = 2\sqrt{\frac{96}{25}}$$

$$\therefore \text{Area of } \triangle PQR = \frac{1}{2} \times PM \times QR = \frac{192\sqrt{6}}{25}$$

$PQRS$ is a rhombus. Therefore,

$$\text{Area of } \triangle QRS = \text{Area of } \triangle PQR = \frac{192\sqrt{6}}{25} \text{ sq. units}$$

18. (2) As $P \equiv (3, 4)$, the equation of QR is

$$3x + 4y = 4$$

(1)

Let $S \equiv (x_1, y_1)$.

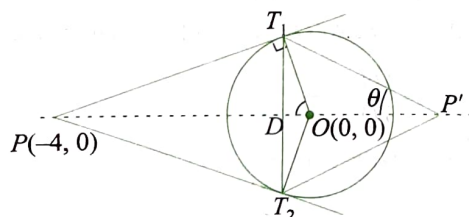
S is the mirror image of P w.r.t. (1). Then,

$$\frac{x_1 - 3}{3} = \frac{y_1 - 4}{4} = \frac{-2(3 \times 3 + 4 \times 4 - 4)}{3^2 + 4^2} = -\frac{42}{25}$$

$$\therefore x_1 = -\frac{51}{25}, y_1 = -\frac{68}{25}$$

$$\text{or } S \equiv \left(-\frac{51}{25}, -\frac{68}{25}\right)$$

9. (1)



$$PT_2 = PT_1 = \sqrt{(-4)^2 + 0^2 - 4} = 2\sqrt{3}$$

The circumcenter of triangle PT_1T_2 is the midpoint of PO as

$$\angle PT_1O = \angle PT_2O = 90^\circ$$

$$\text{so, } \left(\frac{-4+0}{2}, \frac{0+0}{2}\right) \equiv (-2, 0)$$

20. (4) Clearly, $PT_1P'T_2$ is rhombus.

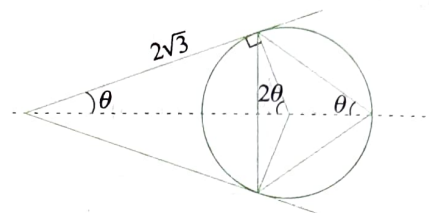
So, area of $\triangle PT_1P'$ and that of $\triangle P'T_1T_2$ are same.

21. (1) Let P' be a point on the circle. Then,

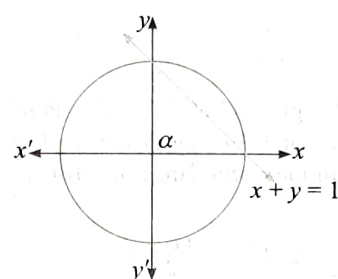
$$3\theta = \frac{\pi}{2}$$

$$\text{or } \theta = \frac{\pi}{6}$$

$$\text{Area of rhombus} = (2\sqrt{3})(2\sqrt{3})\sin \frac{\pi}{3} = 6\sqrt{3}$$



22. (2)



23. (1) Slope of chord = 1

since the chord is a $\pi/3$ chord, its distance from the origin is $\sqrt{3}$.

Let the equation of the chord be $x - y + k = 0$. Then,

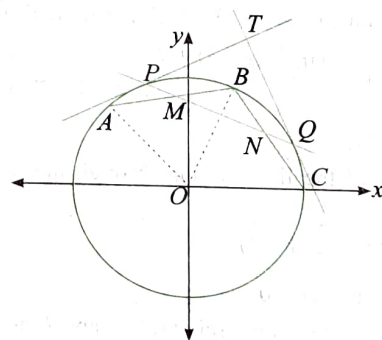
$$\left|\frac{k}{\sqrt{2}}\right| = \sqrt{3}$$

$$\text{i.e., } k = \pm\sqrt{6}$$

24. (1) Radius of circle = 2

$$\text{Distance from the origin} = 2 \cos \frac{\pi}{3} = 1$$

For Problems 25–27



25. (2) From the figure, since $\triangle OAB$ is an equilateral triangle,
 $\angle OAB = 60^\circ$

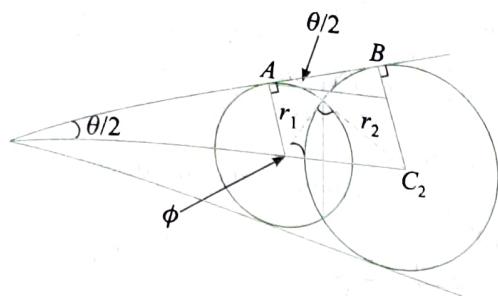
26. (3) Let T be the point of intersection of tangents.

Since $\angle AOC = 120^\circ$, the angle between the tangents is 60° .

27. (3) The locus of the point of intersection of tangents at A and C is a circle whose center is $O(0, 0)$ and radius is

$$OT = a \operatorname{cosec} \frac{\pi}{6} = 2a$$

$$\text{So, the locus is } x^2 + y^2 = 4a^2.$$



We have $\sin \phi = d/r_1$ and $\cos \phi = d/r_2$, (where $2d$ is the length of the common chord). Then,

$$1 = \frac{d^2}{r_1^2} + \frac{d^2}{r_2^2}$$

$$\text{or } d = \frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$$

$$\text{or } 2d = \frac{2r_1 r_2}{\sqrt{r_1^2 + r_2^2}} = \frac{24}{5}, \text{ where } r_1 = 3$$

$$\text{or } d = \frac{6r_2}{\sqrt{9 + r_2^2}} = \frac{24}{5}$$

$$\text{or } r_2 = 4$$

From the figure,

$$\sin \frac{\theta}{2} = \frac{r_2 - r_1}{C_1 C_2}$$

$$\text{where } C_1 C_2^2 = r_1^2 + r_2^2$$

$$\text{or } C_1 C_2 = 5$$

$$\therefore \sin \frac{\theta}{2} = \frac{1}{5}$$

$$\text{or } \cos \frac{\theta}{2} = \frac{\sqrt{24}}{5}$$

$$\therefore \sin \theta = 2 \times \frac{1}{5} \times \frac{\sqrt{24}}{5} = \frac{4\sqrt{6}}{25}$$

$$\text{or } \theta = \sin^{-1} \frac{4\sqrt{6}}{25}$$

$$\text{Also, } AB = \sqrt{C_1 C_2^2 - (r_1 - r_2)^2} = \sqrt{25 - 1} = \sqrt{24}$$

$$31. (2) \frac{r_1}{r_2} = \frac{5}{3} = \frac{AR}{BR} \text{ or } BR = \frac{3 \times 40}{5} = 24 \text{ cm}$$

$$\frac{AB}{BR} = \frac{40 - 24}{24} = \frac{16}{24} = \frac{2}{3}$$

$$32. (3) \frac{r_1}{r_2} = \frac{5}{3}$$

$$\therefore \frac{r_1 + r_2}{r_2} = \frac{8}{3}$$

$$\therefore r_2 = \frac{(40 - 24)3}{8} = 6 \text{ cm}$$

$$33. (4) QR = \sqrt{(BR)^2 - r_2^2} = \sqrt{(24)^2 - 36} = 6\sqrt{15} \text{ cm}$$

34. (2), 35. (4), 36. (1)

$$S_1 \equiv x^2 + y^2 + 4y - 1 = 0$$

$$S_2 \equiv x^2 + y^2 + 6x + y + 8 = 0$$

$$S_3 \equiv x^2 + y^2 - 4x - 4y - 37 = 0$$

$$C_1 \equiv (0, -2), r_1 = \sqrt{5}$$

$$C_2 \equiv \left(-3, \frac{-1}{2}\right), r_2 = \frac{\sqrt{5}}{2}$$

$$C_3 \equiv (2, 2), r_3 = 3\sqrt{5}$$

$$\text{Also, } C_1 C_2 = \sqrt{9 + \frac{9}{4}} = \frac{3\sqrt{5}}{2} = r_1 + r_2$$

So, S_1 and S_2 touch each other externally.

$$C_2 C_3 = \sqrt{25 + \frac{25}{4}} = \frac{5\sqrt{5}}{2} = r_3 - r_2$$

So, S_2 and S_3 touch each other internally.

$$\text{And } C_1 C_3 = \sqrt{4 + 16} = 2\sqrt{5} = r_3 - r_1$$

So, S_1 and S_3 touch each other internally.

The point of contact P_1 divides $C_1 C_2$ internally in the ratio $r_1 : r_2 = 2 : 1$.

$$\Rightarrow P_1 \equiv (-2, -1)$$

The point of contact P_2 divides $C_2 C_3$ externally in the ratio $r_2 : r_3 = 1 : 6$.

$$\Rightarrow P_2 \equiv (-4, -1)$$

The point of contact P_3 divides $C_3 C_1$ externally in the ratio $r_3 : r_1 = 3 : 1$.

$$\Rightarrow P_3 \equiv (-1, -4)$$

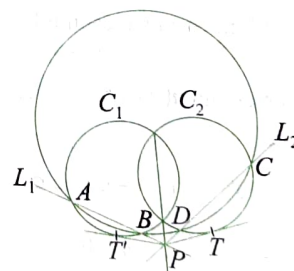
$$\text{Area of } \Delta P_1 P_2 P_3 = \frac{1}{2} \begin{vmatrix} -2 & -1 & 1 \\ -4 & -1 & 1 \\ -1 & -4 & 1 \end{vmatrix} = 3$$

$$\text{And area } \Delta C_1 C_2 C_3 = \frac{1}{2} \begin{vmatrix} 0 & -2 & 1 \\ -3 & \frac{-1}{2} & 1 \\ 2 & 2 & 1 \end{vmatrix} = \frac{15}{2}$$

$$\therefore \frac{\text{area}(\Delta P_1 P_2 P_3)}{\text{area}(\Delta C_1 C_2 C_3)} = \frac{3}{\frac{15}{2}} = 2 : 5$$

Clearly, $P_2(-4, -1)$ and $P_3(-1, -4)$ are images of each other with respect to the line $y = x$.

37. (4)



The given circles are

$$C_1: x^2 + y^2 - 4 = 0$$

$$\text{and } C_2: x^2 + y^2 - 4x - 2y + 1 = 0$$

The given lines are

$$L_1: x + 2y + a = 0$$

$$\text{and } L_2: 12x - 6y - 41 = 0$$

If A, B, C , and D are concyclic, then

$$PA \cdot PB = PC \cdot PD$$

$$\text{or } PT^2 = PT'^2$$

Therefore, point P will lie on the radical axis of circles C_1 and C_2 , i.e., $L_3: 4x + 2y - 5 = 0$.

Now, lines L_1, L_2 , and L_3 are concurrent. Therefore,

$$\begin{vmatrix} 4 & 2 & -5 \\ 1 & 2 & a \\ 12 & -6 & -41 \end{vmatrix} = 0 \text{ or } a = 2$$

Also, for this value of a , the line meets circle (1) at two distinct points.

38. (1) The family of circles through C_1 and L_1 is

$$x^2 + y^2 - 4 + \lambda(x + 2y + 2) = 0$$

Common chord with circle C_2 is

$$(\lambda + 4)x + 2(\lambda + 1)y + 2\lambda - 5 = 0$$

which is the same as $L_2: 12x - 6y - 41 = 0$. Therefore,

$$\lambda = -\frac{8}{5}$$

Hence, the equation of circle is $5x^2 + 5y^2 - 8x - 16y - 36 = 0$.

39. (3), 40. (2)

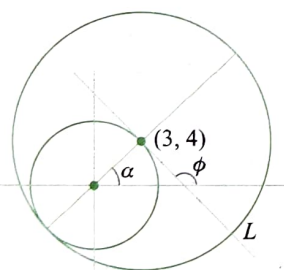
$$A = \left\{ (x, y) \mid \frac{x}{\cos \theta} = \frac{y}{\sin \theta} = 5, \text{ where } \theta \text{ is parameter} \right\}$$

A is set of points which lie on the circle $C_1: x^2 + y^2 = 25$.

$$B = \left\{ (x, y) \mid \frac{x-3}{\cos \phi} = \frac{y-4}{\sin \phi} = r \right\}$$

If ϕ varies and r is fixed, then B is the set of all points which lie on a circle $C_2: (x-3)^2 + (y-4)^2 = r^2$.

And if ϕ is fixed and r varies, then B is the set of all points which lie on the straight line $L: y - 4 = \tan \phi (x - 3)$.



Now, if $A \cap C = A$

Both circles C_1 and C_2 must touch each other internally.

So, R must be 10.

If ϕ is fixed and r varies and $n(A \cap B) = 1$, circle C_1 touches line L .

$$\phi = 90^\circ + \alpha$$

$$\therefore \sec \phi = \sec (90^\circ + \alpha) = -\operatorname{cosec} \alpha = -\frac{5}{4}$$

41. (3) We have $x^2 + y^2 - 2x - 2ay - 8 = 0$

or $(x^2 + y^2 - 2x - 8) - 2ay = 0$, which is of the form $S + \lambda L = 0$.

Solving $S \equiv x^2 + y^2 - 2x - 8 = 0$ and $L \equiv y = 0$, we get

$$x^2 - 2x - 8 = 0$$

$$\Rightarrow (x-4)(x+2) = 0$$

$$\Rightarrow x = -2 \text{ or } 4$$

$$\Rightarrow A \equiv (4, 0), B \equiv (-2, 0)$$

Distance between A and $B = 6$.

42. (4) Let the tangent at points A and B intersect at $P(x_1, y_1)$.

Then AB is chord of contact w.r.t. $P(x_1, y_1)$, which has equation

$$xx_1 + yy_1 - (x + x_1) - a(y + y_1) - 8 = 0$$

$$\Rightarrow x(x_1 - 1) + y(y_1 - a) - (x_1 + ay_1 + 8) = 0 \quad (1)$$

Also, equation of AB is x -axis i.e.,

$$y = 0 \quad (2)$$

On comparing ratio of coefficients of equations (1) and (2), we get

$$\frac{x_1 - 1}{0} = \frac{y_1 - a}{1} = \frac{x_1 + ay_1 + 8}{0} = k \text{ (say)}$$

$$\Rightarrow x_1 - 1 = 0, y_1 = a + k \text{ and } x_1 + ay_1 + 8 = 0$$

$$\Rightarrow x_1 = 1 \text{ and } 9 + ay_1 = 0$$

$$\therefore y_1 = -\frac{9}{a}$$

Also, (x_1, y_1) lies on line $x + 2y + 5 = 0$.

$$\therefore x_1 + 2y_1 + 5 = 0$$

$$\text{or } 1 - \frac{18}{a} + 5 = 0$$

$$\therefore a = 3$$

So, required equation of circle $S = 0$ is $x^2 + y^2 - 2x - 6y - 8 = 0$.

Now, radius of circle $S = 0$ is $r = \sqrt{1 + 9 + 8} = 3\sqrt{2}$.

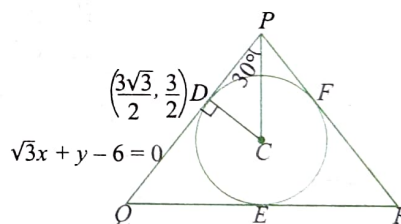
43. (4) Circle $x^2 + y^2 - 10x + 2y + c = 0$ is orthogonal to $x^2 + y^2 - 2x - 6y - 8 = 0$.

So, by applying condition of orthogonal intersection, we get

$$2(-5)(-1) + 2(1)(-3) = c - 8$$

$$\therefore c = 12$$

44. (4)



$$\text{Equation of } PQ \text{ is } L_1: \sqrt{3}x + y - 6 = 0 \quad (1)$$

Slope of PQ is $-\sqrt{3}$

$$\therefore \text{Slope of } CD \text{ is } \frac{1}{\sqrt{3}}$$

$\therefore$ Equation of line CD in parametric form is

$$\frac{x - \frac{3\sqrt{3}}{2}}{\cos 30^\circ} = \frac{y - \frac{3}{2}}{\sin 30^\circ} = r$$

$$\text{or } \frac{x - \frac{3\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = \frac{y - \frac{3}{2}}{\frac{1}{2}} = r,$$

For centre C , $r = \pm 1$

Two possible coordinates of centre are $(2\sqrt{3}, 2)$, $(\sqrt{3}, 1)$.

According to the question, center lies on the same side where origin lies with respect to line PQ .

$$\text{Now } L_1(0, 0) = -6 < 0$$

$$\text{And } L_1(\sqrt{3}, 1) = 3 + 1 - 6 < 0$$

So Centre C must be $(\sqrt{3}, 1)$

Hence, equation of the circle is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

45. (1) By simple geometry $PD = \sqrt{3}$ (ΔPQR is equilateral)
Equation of PQ in parametric form is

$$\frac{x - \frac{3\sqrt{3}}{2}}{\cos 120^\circ} = \frac{y - \frac{3}{2}}{\sin 120^\circ} = r$$

or

$$\frac{x - \frac{3\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{y - \frac{3}{2}}{\frac{\sqrt{3}}{2}} = r$$

Now points at distance $\sqrt{3}$ from point D on line PQ are given by

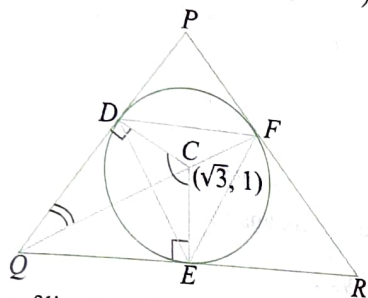
$$\left(\frac{3\sqrt{3}}{2} \pm \sqrt{3} \left(-\frac{1}{2} \right), \frac{3}{2} \pm \sqrt{3} \left(\frac{\sqrt{3}}{2} \right) \right) \text{ or } (2\sqrt{3}, 0) \text{ and } (\sqrt{3}, 3)$$

Point C divides the join of P and E in the ratio 2 : 1.

Similarly, C divides join of Q and F in the ratio 2 : 1.

Thus, co-ordinates of E and F are $\left(\frac{\sqrt{3}}{2}, \frac{3}{2} \right)$ and $(\sqrt{3}, 0)$.

46. (4)



Equation of line PR which is parallel to DE and passes through F is $(y - 0) = 0(x - \sqrt{3}) \Rightarrow y = 0$.

Similarly, equation of line QR which is parallel to DF and passes through the point E is

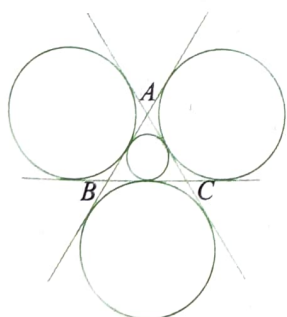
$$\left(y - \frac{3}{2} \right) = \left(\frac{\frac{3}{2} - 0}{\frac{3\sqrt{3}}{2} - \sqrt{3}} \right) \left(x - \frac{\sqrt{3}}{2} \right)$$

$$\Rightarrow y = \sqrt{3}x$$

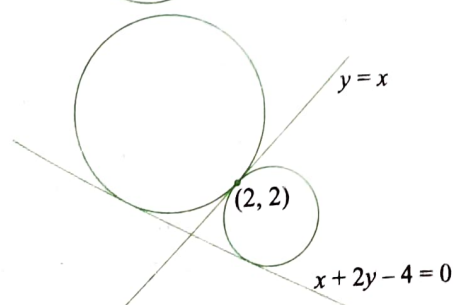
Matrix Match Type

1. $a \rightarrow r$; $b \rightarrow q$; $c \rightarrow q$; $d \rightarrow p$

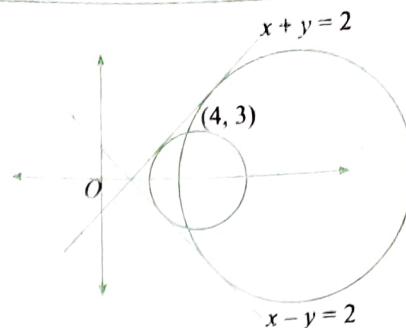
a.



b.



c.



d. Obviously one (see theory).

2. $a \rightarrow p, r, s$; $b \rightarrow r, s$; $c \rightarrow q, s$; $d \rightarrow p, s$

a. The point $(-g, -f)$ lies in the first quadrant. Then $g < 0$ and $f < 0$. Also, the x - and y -axes must not cut the circle.

Solving the circle and the x -axis, we have $x^2 + 2gx + c = 0$, which must have imaginary roots. So, $g^2 - c < 0$. Then c must be positive. Also, $f^2 - c < 0$.

b. If the circle lies above the x -axis, then $x^2 + 2gx + c = 0$ must have imaginary roots. Therefore, $g^2 - c < 0$, and $c > 0$.

c. The point $(-g, -f)$ lies in the third or fourth quadrant. Then $g > 0$. Also, the y -axis must not cut the circle.

Solving the circle and the y -axis, we have $y^2 + 2gy + c = 0$, which must have imaginary roots. Then, $f^2 - c < 0$. So, c must be positive.

d. $x^2 + 2gx + c = 0$ must have equal roots. Then $g^2 = c$. Hence, $c > 0$.

Also, $-g > 0$ or $g < 0$.

3. $a \rightarrow r$; $b \rightarrow s$; $c \rightarrow q$; $d \rightarrow p$

a. Since $(2, 3)$ lies inside the circle, such chord is bisected at $(2, 3)$, which has equation

$$y - 3 = -(x - 2)$$

$$\text{or } x + y - 5 = 0 \text{ or } a = b = 1$$

b. Let P be the point (α, β) . Then $\alpha^2 + \beta^2 + 2\alpha + 2\beta = 0$.

The midpoint of OP is $(\alpha/2, \beta/2)$.

Therefore, the locus of $(\alpha/2, \beta/2)$ is

$$4x^2 + 4y^2 + 4x + 4y = 0$$

$$\text{i.e., } x^2 + y^2 + x + y = 0$$

$$\therefore 2g = 1, 2f = 1$$

$$\text{and } g + f = 1$$

c. The centers of the circles are $(1, 2)$ and $(5, -6)$.

The equation of C_1C_2 is

$$y - 2 = -\frac{8}{4}(x - 1)$$

$$\text{i.e., } 2x + y - 4 = 0$$

The equation of the radical axis is

$$8x - 16y - 56 = 0$$

$$\text{i.e., } x - 2y - 7 = 0$$

The point of intersection is $(3, -2)$.

d. If θ is the angle between the tangents, then

$$\frac{\theta}{2} = \frac{\text{Radius}}{\text{Distance between } (3\sqrt{3}, 3) \text{ and } (0, 0)} = \frac{1}{2}$$

$$\text{or } \frac{\theta}{2} = \frac{\pi}{6} \text{ or } \theta = \frac{\pi}{3} \text{ or } 2\sqrt{3} \tan \theta = 6$$

4. $a \rightarrow r$; $b \rightarrow p, q$; $c \rightarrow q, r$; $d \rightarrow p, s$

- a. The radical axis of $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2x + b = 0$ is $(a_1 - a_2)x = 0$ or $x = 0$. It must touch both the circles.

Solving it with one of the circles, we get

$$y^2 + b = 0 \text{ or } b \leq 0$$

- b. The radical axis of $x^2 + y^2 + 2a_1x + b = 0$ and $x^2 + y^2 + 2a_2y + b = 0$ is $a_1x - a_2y = 0$.

Solving it with one the circles, we have

$$x^2 + \left(\frac{a_1}{a_2}\right)^2 x^2 + 2a_1x + b = 0$$

This equation must have equal roots. Hence,

$$4a_1^2 - 4b \left[1 + \left(\frac{a_1^2}{a_2^2}\right) \right] = 0$$

$$a_1^2 - b \left[1 + \frac{a_1^2}{a_2^2} \right] = 0$$

Options p and q satisfy this condition.

- c. If the straight line $a_1x - by + b^2 = 0$ touches the circle $x^2 + y^2 = 2a_2x + by$, then

$$\frac{\left| a_1 \frac{a_2}{2} - b \frac{b}{2} + b^2 \right|}{\sqrt{a_1^2 + b^2}} = \sqrt{\frac{a_2^2}{4} + \frac{b^2}{4}}$$

$$\frac{|a_1a_2 + b^2|}{\sqrt{a_1^2 + b^2}} = \sqrt{a_2^2 + b^2}$$

$$\text{or } a_1^2a_2^2 + 2b^2a_1a_2 + b^4 = a_1^2a_2^2 + a_1^2b^2 + a_2^2b^2 + b^4$$

$$\text{i.e., } b^2 = 0 \text{ or } 2a_1a_2 = a_1^2 + a_2^2$$

Options q and r satisfy this condition.

- d. The line $3x + 4y - 4 = 0$ touches the circle $(x - a_1)^2 + (y - a_2)^2 = b^2$. Then,

$$\frac{|3a_1 + 4a_2 - 4|}{5} = b$$

5. (3)

- a. Let the length of the common chord be $2a$. Then,

$$\sqrt{9 - a^2} + \sqrt{16 - a^2} = 5$$

$$\text{or } \sqrt{16 - a^2} = 5 - \sqrt{9 - a^2}$$

$$\text{or } 16 - a^2 = 25 + 9 - a^2 - 10\sqrt{9 - a^2}$$

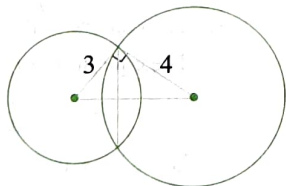
$$\text{or } 10\sqrt{9 - a^2} = 18$$

$$\text{or } 100(9 - a^2) = 324$$

$$\text{i.e., } 100a^2 = 576$$

$$\text{or } a = \sqrt{\frac{576}{100}} = \frac{24}{10}$$

$$\therefore 2a = \frac{24}{5} = \frac{k}{5} \text{ or } k = 24$$



- b. The equation of common chord is $6x + 4y + p + q = 0$.

The common chord passes through the center $(-2, -6)$ of the circle $x^2 + y^2 + 4x + 12y + p = 0$. Therefore,

$$p + q = 36$$

- c. The equation of the circle is $2x^2 + 2y^2 - 2\sqrt{2}x - y = 0$.

Let $(\alpha, 0)$ be the midpoint of a chord. Then, the equation of the chord is

$$2\alpha x - \sqrt{2}(x + \alpha) - \frac{1}{2}(y + 0) = 2\alpha^2 - 2\sqrt{2}\alpha$$

Since it passes through the point $(\sqrt{2}, 1/2)$, we have

$$2\sqrt{2}\alpha - \sqrt{2}(\sqrt{2} + \alpha) - \frac{1}{4} = 2\alpha^2 - 2\sqrt{2}\alpha$$

$$\text{i.e., } 8\alpha^2 - 12\sqrt{2}\alpha + 9 = 0$$

$$\text{i.e., } (2\sqrt{2}\alpha - 3)^2 = 0$$

$$\text{i.e., } \alpha = \frac{3}{2\sqrt{2}}$$

Therefore, the number of chords is 1.

- d. Midpoint of $AB \equiv (1, 4)$

The equation perpendicular bisector of AB is $x = 1$.

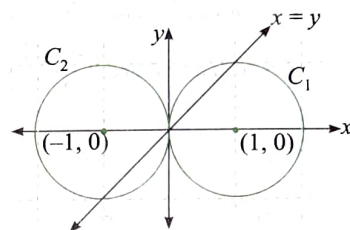
A diameter of the circle is $4y = x + 7$.

Therefore, the center of the circle is $(1, 2)$.

Hence, the sides of the rectangle are 8 and 4. Therefore,

Area = 32

6. (4)



(λ, λ) lies on the line $y = x$.

From the diagram, answer is (4)

Numerical Value Type

1. (6) Clearly, the locus of the point of intersection of lines is

$$(x - 5)(x - 3) + (y - 2)(y + 4) = 0$$

$$\text{or } x^2 + y^2 - 8x + 2y + 7 = 0$$

$$\text{Hence, } |f + g| = |2 + (-8)| = 6$$

2. (6) $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$

$$\text{or } (x^2 + y^2 - 2x - 8) - 2\lambda y = 0$$

which is of the form of $S + \lambda L = 0$.

All the circles pass through the point of intersection of the circle $x^2 + y^2 - 2x - 8 = 0$ and $y = 0$.

Solving, we get

$$x^2 - 2x + 8 = 0 \text{ or } (x - 4)(x + 2) = 0$$

$$\therefore A \equiv (4, 0) \text{ and } B \equiv (-2, 0)$$

So, the distance between A and B is 6.

3. (8) We have $\tan^4 x + \cot^4 x + 2 = 4 \sin^2 y$

$$\text{or } (\tan^2 x - \cot^2 x)^2 + 4 = 4 \sin^2 y$$

Now, $\text{LHS} \geq 4$ and $\text{RHS} \leq 4$. Therefore,

$$\tan^2 x = 1 \text{ and } \sin^2 y = 1 \text{ or } \tan x = \pm 1 \text{ and } \sin y = \pm 1$$

But $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$.

Therefore, the acceptable values of x are $\pm\pi/4$ and $\pm 3\pi/4$. The acceptable values of y are $\pm\pi/2$.

Hence, the number of points $P(x, y)$ are 8.

4. (1) Let $x + 5 = 14 \cos \theta$ and $y - 12 = 14 \sin \theta$. Then

$$x^2 + y^2 = (14 \cos \theta - 5)^2 + (14 \sin \theta + 12)^2$$

$$= 196 + 25 + 144 + 28(12 \sin \theta - 5 \cos \theta)$$

$$= 365 + 28(12 \sin \theta - 5 \cos \theta)$$

$$\therefore \sqrt{x^2 + y^2} \Big|_{\min} = \sqrt{365 - 28 \times 13} = \sqrt{365 - 364} = 1$$

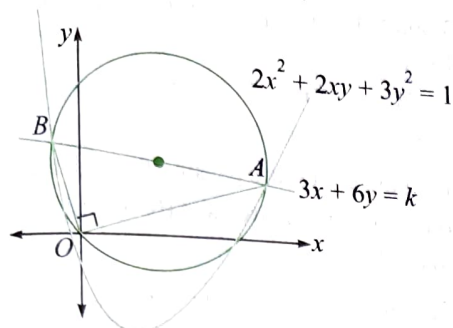
$$5. (9) \quad 3x + 6y = k$$

$$\text{or} \quad \frac{3x + 6y}{k} = 1$$

$$\text{Also, } 2x^2 + 2xy + 3y^2 - 1 = 0$$

Now, homogenizing (2) with the help of (1), we get

$$2x^2 + 2xy + 3y^2 - \left(\frac{3x + 6y}{k}\right)^2 = 0$$



$$\text{or} \quad k^2(2x^2 + 2xy + 3y^2) - (3x + 6y)^2 = 0 \quad (3)$$

This is the equation of the pair of straight lines OA and OB .

Now, AB is diameter. Then OA and OB are perpendicular. So,

Coefficient of x^2 + Coefficient of y^2 = 0

$$\text{or} \quad (2k^2 - 9) + (3k^2 - 36) = 0$$

$$\text{or} \quad 5k^2 = 45$$

$$\text{or} \quad k^2 = 9$$

6. (0) Since both the circles are symmetric about the x -axis, the sum of slopes of tangents is 0.

7. (4) The radius of the given circle $x^2 + y^2 + 4x - 2\sqrt{2}y + c = 0$

$$\text{is } \sqrt{4 + 2 - c} = \sqrt{6 - c} = a \text{ (let).}$$

$$\text{Now, radius of circle } S_1 = \frac{a}{\sqrt{2}}$$

$$\text{Radius of circle } S_2 = \frac{a}{2}$$

and so on.

$$\text{Also, } a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \dots + \infty = 2 \text{ (Given)}$$

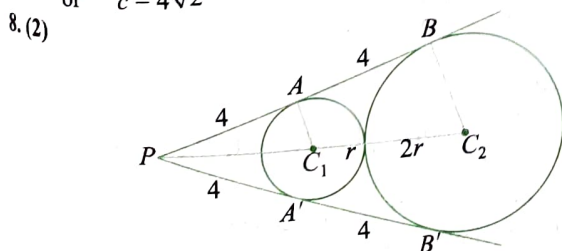
$$\text{Then } a \left(\frac{1}{1 - 1/\sqrt{2}} \right) = 2$$

$$\text{or} \quad \frac{a\sqrt{2}}{\sqrt{2} - 1} = 2$$

$$\text{or} \quad a = 2 - \sqrt{2} = \sqrt{6 - c}$$

$$\text{or} \quad 4 + 2 - 4\sqrt{2} = 6 - c$$

$$\text{or} \quad c = 4\sqrt{2}$$



$$\frac{AC_1}{PA} = \frac{BC_2}{PB}$$

$$\text{or} \quad BC_2 = 2AC_1$$

$$PC_1 = \sqrt{16 + r^2}$$

$$\text{and } PC_2 = \sqrt{64 + 4r^2} = 2\sqrt{16 + r^2}$$

$$PC_2 - PC_1 = 3r$$

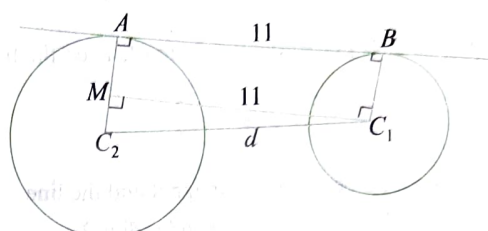
$$\text{or} \quad 2\sqrt{16 + r^2} - \sqrt{16 + r^2} = 3r$$

$$\text{or} \quad \sqrt{16 + r^2} = 3r$$

$$\text{or} \quad 16 + r^2 = 9r^2$$

$$\text{or} \quad r^2 = 2$$

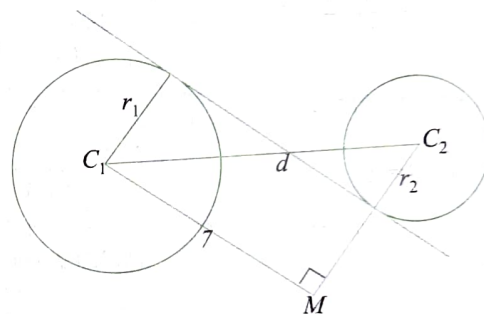
9. (18)



From the figure,

$$d^2 = 11^2 + (r_1 - r_2)^2$$

(1)



From the figure,

$$d^2 = 7^2 + (r_1 + r_2)^2$$

(2)

From (1) and (2),

$$4r_1r_2 = 72 \text{ or } r_1r_2 = 18$$

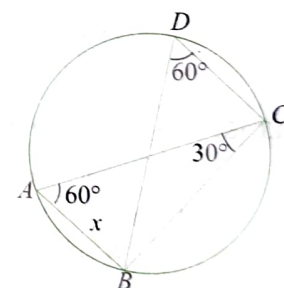
10. (1)

$$\angle A = 60^\circ = \angle D$$

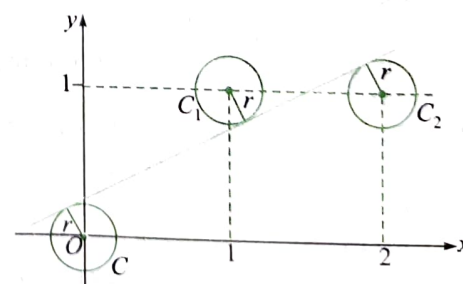
$$AC = 2 \text{ (Given)}$$

$$\angle ABC = 90^\circ$$

$$\text{or} \quad x = 1$$



11. (5)



The equation of the line joining the origin and the center of circle C_2 , (2, 1), is

$$y = \frac{x}{2}$$

$$\text{or} \quad x - 2y = 0$$

Let the equation of common tangent be

$$x - 2y + c = 0$$

$\therefore$ Perpendicular distance from $(0, 0)$ on this line

$$= \text{Perpendicular distance from } (1, 1)$$

$$\text{or } \left| \frac{c}{\sqrt{5}} \right| = \left| \frac{c-1}{\sqrt{5}} \right|$$

$$\text{or } c = 1 - c \text{ or } c = \frac{1}{2}$$

The equation of common tangent is

$$x - 2y + \frac{1}{2} = 0 \text{ or } 2x - 4y + 1 = 0 \quad (2)$$

The length of perpendicular from $(2, 1)$ on the line (2) is

$$r = \left| \frac{4 - 4 + 1}{\sqrt{20}} \right| = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

12. (3) θ is the angle between the tangent and the line

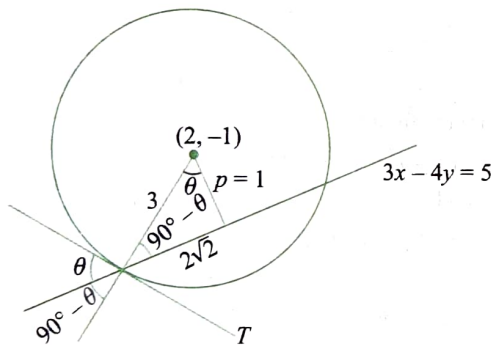
The circle has center $(2, -1)$ and radius 3.

The length of perpendicular from the center on $3x - 4y = 5$ is

$$p = \left| \frac{6 + 4 - 5}{5} \right| = 1$$

$$\text{or } \sin(90^\circ - \theta) = \frac{1}{3}$$

$$\text{or } \cos \theta = \frac{1}{3}$$



13. (5) The equation of the given circle is

$$x^2 + y^2 - 6x - 2py + 17 = 0$$

$$\text{or } (x-3)^2 + (y-p)^2 = (p^2 - 8) \quad (1)$$

Also, $(0, 0)$ lies outside the circle.

The equation of director circle of $S = 0$ will be

$$(x-3)^2 + (y-p)^2 = 2(p^2 - 8) \quad (2)$$

Tangents drawn from $(0, 0)$ to circle (1) are perpendicular to each other.

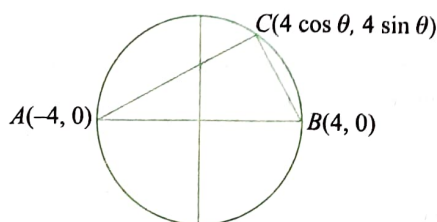
Therefore, $(0, 0)$ must lie on director circle. Hence,

$$(0-3)^2 + (0-p)^2 = 2(p^2 - 8)$$

$$\text{or } p^2 = 25$$

$$\text{or } p = \pm 5$$

14. (62)



Required area

$$A = \frac{1}{2} \times 8 \times |4 \sin \theta| = |16 \sin \theta|$$

Now, the area is an integer. Then the possible values of $\sin \theta$ are

$$\frac{1}{16}, \frac{2}{16}, \dots, \frac{15}{16}$$

i.e., 15 points in each quadrant or $60 + 2$ more with $\sin \theta = 1$. Therefore, $N = 62$.

$$15. (1) x^2 + y^2 + (3 + \sin \beta)x + (2 \cos \alpha)y = 0 \quad (1)$$

$$x^2 + y^2 + (2 \cos \alpha)x + 2cy = 0 \quad (2)$$

Since both the circles are passing through the origin $(0, 0)$, the equation of tangent to circles at $(0, 0)$ will be same. Tangent at $(0, 0)$ to circle (1) is,

$$(3 + \sin \beta)x + (2 \cos \alpha)y = 0 \quad (3)$$

Tangent at $(0, 0)$ to circle (2) is

$$(2 \cos \alpha)x + 2cy = 0$$

Therefore, (1) and (2) must be identical.

Comparing (1) and (2), we get

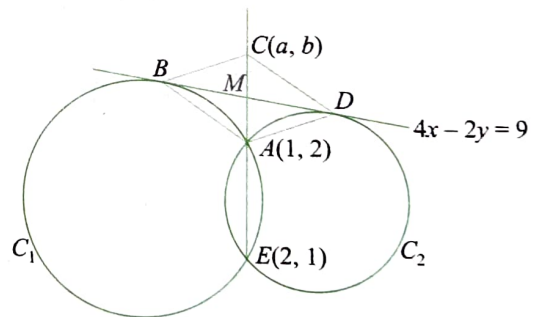
$$\frac{3 + \sin \beta}{2 \cos \alpha} = \frac{2 \cos \alpha}{2c}$$

$$\text{or } c = \frac{2 \cos^2 \alpha}{3 + \sin \beta}$$

$$\therefore c_{\max} = 1 \text{ when } \sin \beta = -1 \text{ and } \alpha = 0$$

16. (4) The radical axis bisects the common tangent BD . Hence, M is the midpoint of BD .

Let $C \equiv (a, b)$.



Now, $C(a, b)$ lies on common chord AE which is $y - 2 = -1(x - 1)$ or $x + y = 3$. Therefore,

$$a + b = 3 \quad (1)$$

Also, $M\left(\frac{a+1}{2}, \frac{b+2}{2}\right)$ lies on $4x - 2y = 9$. Therefore,

$$4\left(\frac{a+1}{2}\right) - 2\left(\frac{b+2}{2}\right) = 9$$

$$\text{or } 2a + 2 - b - 2 = 9$$

$$\text{or } 2a - b = 9 \quad (2)$$

Solving (1) and (2), $a = 4$ and $b = -1$. Therefore,

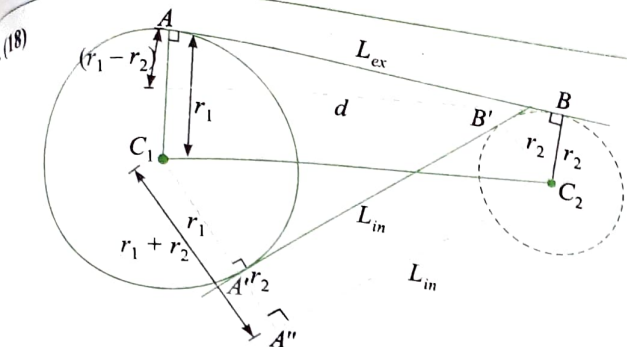
$$\therefore |ab| = 4$$

17. (4) Let r be the radius of the required circle.

Now, if two circles touch each other, then

Distance between their centers $= |r \pm 2| = 5$ (Given)

$$\therefore r = 3, 7$$



From the above figure,

$$L_{in} = \sqrt{d^2 - (r_1 + r_2)^2} = 7$$

$$\text{and } L_{ex} = \sqrt{d^2 - (r_1 - r_2)^2} = 11$$

Squaring and subtracting, we get $r_1 r_2 = 18$.

Archives

Main

Correct Answer Type

1. (2) $x^2 + y^2 + 3x + 7y + 2p - 5 + \lambda(x^2 + y^2 + 2x + 2y - p^2) = 0$,
 $\lambda \neq -1$ passes through point of intersection of given circles.

Since it passes through (1, 1),

$$7 - 2p + \lambda(6 - p^2) = 0$$

$$\Rightarrow 7 - 2p + 6\lambda - \lambda p^2 = 0$$

If $\lambda = -1$, then

$$7 - 2p - 6 + p^2 = 0$$

$$\Rightarrow p^2 - 2p + 1 = 0$$

$$\therefore p = 1$$

If $\lambda \neq -1$ then $p \neq 1$.

Therefore, all values of p are possible except $p = 1$.

2. (2) $P(1, 0); Q(-1, 0)$

Let $A = (x, y)$

$$\frac{AP}{AQ} = \frac{BP}{BQ} = \frac{CP}{CQ} = \frac{1}{3}$$

$$\Rightarrow 3AP = AQ$$

$$\Rightarrow 9AP^2 = AQ^2$$

$$\Rightarrow 9(x-1)^2 + 9y^2 = (x+1)^2 + y^2$$

$$\Rightarrow x^2 + y^2 - (5/2)x + 1 = 0$$

... (1)

Therefore A lies on a circle.

Similarly, B and C also lie on the same circle.

Hence, the circumcentre of ABC is centre of circle (1) which is $(5/4, 0)$.

3. (3) The circle is

$$x^2 + y^2 - 4x - 8y - 5 = 0$$

Its centre is $(2, 4)$ and radius is $\sqrt{4 + 16 + 5} = 5$

If the circle intersects the line $3x - 4y = m$ at two distinct points, then the length of the perpendicular from the centre is less than the radius, i.e.,

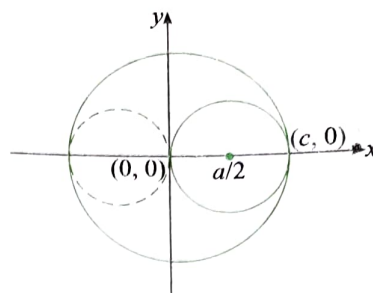
$$\frac{|6 - 16 - m|}{5} < 5$$

$$\Rightarrow |10 + m| < 25$$

$$\Rightarrow -25 < m + 10 < 25$$

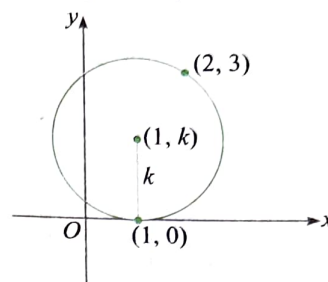
$$\Rightarrow -35 < m < 15$$

4. (3)



The figure is self-explanatory. Clearly $c = |a|$

5. (1)



Circle touches x -axis at $(1, 0)$.

So, center of circle is $(1, k)$ and radius is k .

Hence, equation of circle is:

$$(x-1)^2 + (y-k)^2 = k^2$$

It passes through the point $(2, 3)$, so

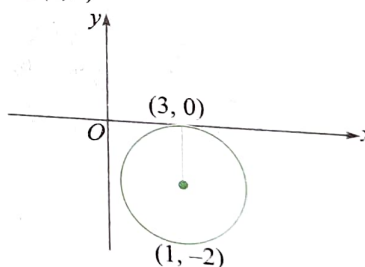
$$(2-1)^2 + (3-k)^2 = k^2$$

$$\therefore 1 + 9 - 6k + k^2 = k^2$$

$$\therefore k = 5/3$$

$$\therefore \text{diameter} = 2k = 10/3$$

6. (3) Let centre $C(3, k)$



As circle touches X -axis

$$\Rightarrow r = k$$

So, circle is $(x-3)^2 + (y-k)^2 = k^2$

Given that it passes through $(1, -2)$.

$$4 + (k+2)^2 = k^2$$

$$\Rightarrow 4 + k^2 + 4k + 4 = k^2$$

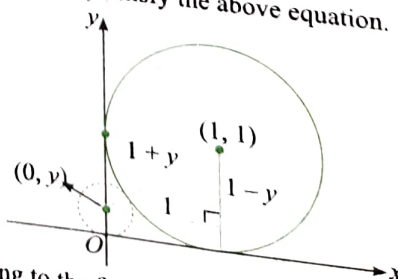
$$\Rightarrow 4k = -8$$

$$\Rightarrow k = -2$$

Circle is $(x-3)^2 + (y+2)^2 = 4$

Obviously $(5, -2)$ satisfy the above equation.

7. (4)

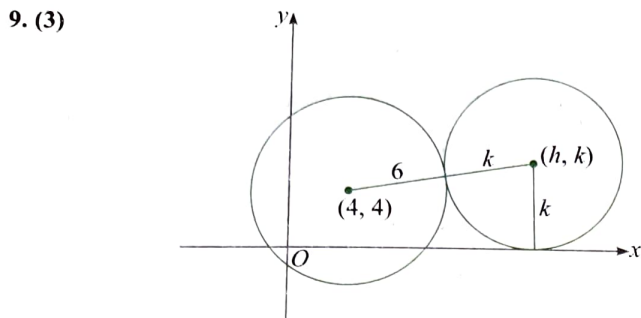


According to the figure

$$(1+y)^2 = (1-y)^2 + 1 \quad (y > 0)$$

$$\Rightarrow y = \frac{1}{4}$$

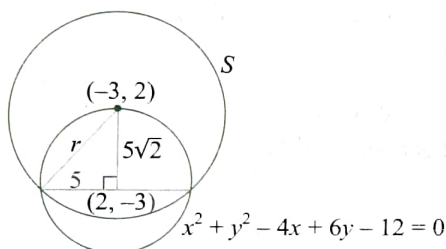
8. (3) $C_1 \equiv (2, 3), r_1 = 5$
 $C_2 \equiv (-3, -9), r_2 = 8$
 $C_1 C_2 = \sqrt{25 + 144} = 13$
 $C_1 C_2 = r_1 + r_2$
 $\Rightarrow$ circles touch externally.
 $\Rightarrow$ 3 common tangents



Let (h, k) be the centre of the circle which touch x -axis and $x^2 + y^2 - 8x - 8y - 4 = 0$ externally.

- $\Rightarrow$ Radius of that circle is $|k|$
 $\Rightarrow (h-4)^2 + (k-4)^2 = (|k| + 6)^2$
 $\Rightarrow x^2 - 8x - 20y - 4 = 0$ if $y \geq 0$
and $x^2 - 8x + 4y - 4 = 0$ if $y < 0$
 $\Rightarrow$ The curve is parabola

10. (1)



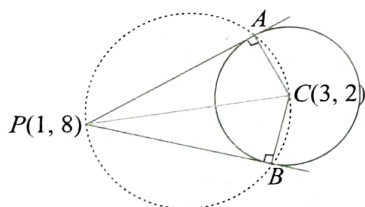
Let ' r ' be the radius of circle S

$$\Rightarrow r = 5\sqrt{3}$$

JEE Advanced

Single Correct Answer Type

1. (2) The center of the circle is $C(3, 2)$.



Since CA and CB are perpendicular to PA and PB , CP is the diameter of the circumcircle of triangle PAB . Its equation is

$$(x-3)(x-1) + (y-2)(y-8) = 0$$

or $x^2 + y^2 - 4x - 10y + 19 = 0$

2. (4) Equation of circle touching the y -axis at $(0, 2)$ is

$$(x-0)^2 + (y-2)^2 + \lambda x = 0$$

It passes through $(-1, 0)$. Therefore,

$$1 + 4 - \lambda = 0 \text{ or } \lambda = 5$$

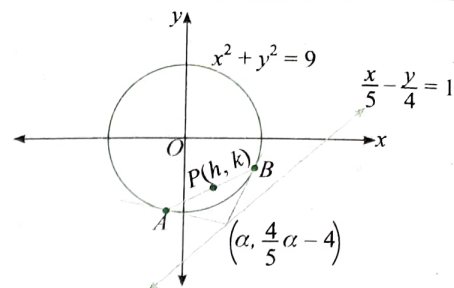
$\therefore$ Equation of circle is

$$x^2 + y^2 + 5x - 4y + 4 = 0$$

Putting $y = 0$, we get $x = -1, -4$.

Therefore, the circle passes through $(-4, 0)$.

3. (1)



The equation of the chord AB bisected at $P(h, k)$ is

$$hx + ky = h^2 + k^2$$

Let any point on the given line be

$$\left(\alpha, \frac{4}{5} \alpha - 4 \right)$$

The equation of the chord of contact AB is

$$\alpha x + \left(\frac{4}{5} \alpha - 4 \right) y = 9$$

Comparing (1) and (2), we get

$$\frac{h}{\alpha} = \frac{k}{(4/5)\alpha - 4} = \frac{h^2 + k^2}{9}$$

$$\therefore \alpha = \frac{20h}{4h - 5k} \quad (\text{From 1}^{\text{st}} \text{ and 2}^{\text{nd}} \text{ ratio})$$

$$\therefore \frac{h(4h - 5k)}{20h} = \frac{h^2 + k^2}{9}$$

$$\text{or } 20(h^2 + k^2) = 9(4h - 5k)$$

$$\text{or } 20(x^2 + y^2) - 36x + 45y = 0$$

Multiple Correct Answers Type

1. (1), (3)

The figure shows circles touching the x -axis at $A(3, 0)$ and having intercept $BD = 2\sqrt{7}$ on the y -axis.

From the figure, the abscissa of center is 3.

Also, $BC = \text{Radius of the circle}$

$$= \sqrt{CM^2 + BM^2} = \sqrt{9 + 7} = 4.$$

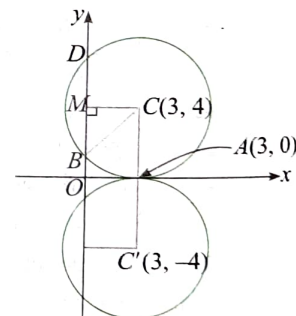
Therefore, $AC = 4$

So, the centers of the circles are $C(3, 4)$ and $C'(3, -4)$.

Hence, the equations of circles are

$$(x-3)^2 + (y \pm 4)^2 = 16$$

$$\text{or } x^2 + y^2 - 6x \pm 8y + 9 = 0$$



2. (2), (3)

Given circles

$$S_1 : x^2 + y^2 - 2x - 15 = 0$$

$$\text{and } S_2 : x^2 + y^2 - 1 = 0$$

Center of the circle which intersects above two circles orthogonally lies on the radical axis of the circles which is $S_1 - S_2 = 0$ or $x + 7 = 0$

Let the centre of the required circle be $C(-7, k)$.

Circle passes through the point $A(0, 1)$.

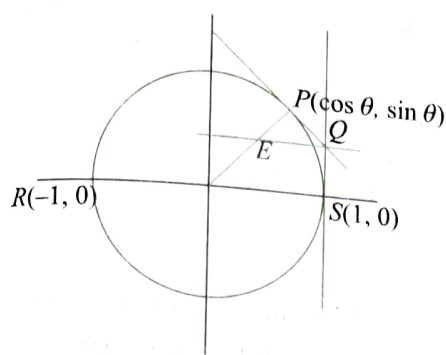
$$\therefore \text{radius, } r = \sqrt{7^2 + (k-1)^2}$$

Also, radius = length of the tangent from C to the any one of the given circles.

$$\therefore r = \sqrt{7^2 + k^2 - 1}$$

On comparing, we get $7^2 + (k-1)^2 = 7^2 + k^2 - 1$
 or $-2k + 1 = -1$
 or $k = 1$
 or $r = 7$

3. (1), (3)



Let point P be $(\cos \theta, \sin \theta)$.
 Tangent at P is $x \cos \theta + y \sin \theta = 1$.

$$\therefore Q \equiv \left(1, \frac{1 - \cos \theta}{\sin \theta}\right)$$

Normal at P , $y = \tan \theta x$

$$\text{Equation of } QE, y = \frac{1 - \cos \theta}{\sin \theta}$$

Point E is point of intersection of lines (i) and (ii).

Eliminating θ from (i) and (ii), we get locus of point E .

$$\text{From (i), } \tan \theta = \frac{y}{x}$$

From eq. (ii), we get

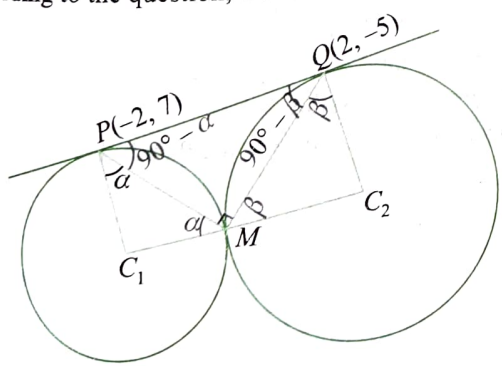
$$y = \frac{1 - \frac{x}{\sqrt{x^2 + y^2}}}{\frac{y}{\sqrt{x^2 + y^2}}} = \frac{\sqrt{x^2 + y^2} - x}{y}$$

$$\text{or } y^2 + x = \sqrt{x^2 + y^2}$$

Clearly option (1) and (3) satisfy the above equation.

4. (2), (4)

According to the question, we have the following figure:



Clearly, $\angle PMQ = \frac{\pi}{2}$.

Hence, locus of M is

$$(x+2)(x-2) + (y-7)(y+5) = 0 \quad (i)$$

$$\text{or } E_1 \equiv x^2 + y^2 - 2y - 39 = 0$$

Clearly, E_2 is the locus of mid-points of chords of E_1 passing through $(1, 1)$.

Let the mid-point of chord of E_1 be (h, k) .

Its equation is

$$h + k - (1 + k) - 39 = h^2 + k^2 - 2k - 39$$

$$\text{or } h^2 + k^2 - 2k - h + 1 = 0$$

$$\text{Hence, } E_2 \equiv x^2 + y^2 - x - 2y + 1 = 0$$

Option (A) is incorrect, although, it satisfies eq. (i) otherwise the line T would touch the second circle at two points.
 Also, $(4/5, 7/5)$ satisfies eq. (ii) but again in this case one end of the chord would be $(-2, 7)$ which is not included in E_1 . Therefore, $(4/5, 7/5)$ does not lie in E_2 .
 $(1/2, 1)$ does not satisfy eq. (ii), therefore it does not lie in E_2 .
 $(0, 3/2)$ does not satisfy eq. (i), so it does not lie in E_1 .

Linked Comprehension Type

1. (1) The equation of tangent at $P(\sqrt{3}, 1)$ is $\sqrt{3}x + y = 4$

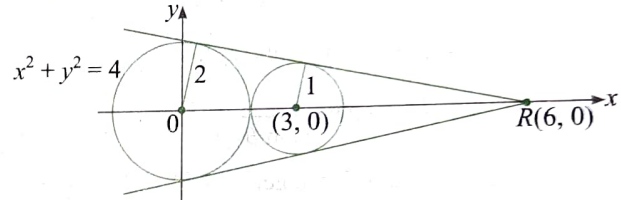
The slope of line perpendicular to the above tangent is $1/\sqrt{3}$.
 So, the equations of tangents with slope $1/\sqrt{3}$ to $(x-3)^2 + y^2 = 1$ will be

$$y = \frac{1}{\sqrt{3}}(x-3) \pm 1\sqrt{1 + \frac{1}{3}}$$

$$\text{or } \sqrt{3}y = x - 3 \pm (2)$$

$$\text{i.e., } \sqrt{3}y = x - 1 \text{ or } \sqrt{3}y = x - 5$$

2. (4) The point of intersection of direct common tangents is $(6, 0)$.



So, let the equation of common tangent be

$$y - 0 = m(x - 6)$$

As it touches $x^2 + y^2 = 4$, we have

$$\left| \frac{0 - 0 - 6m}{\sqrt{1 + m^2}} \right| = 2$$

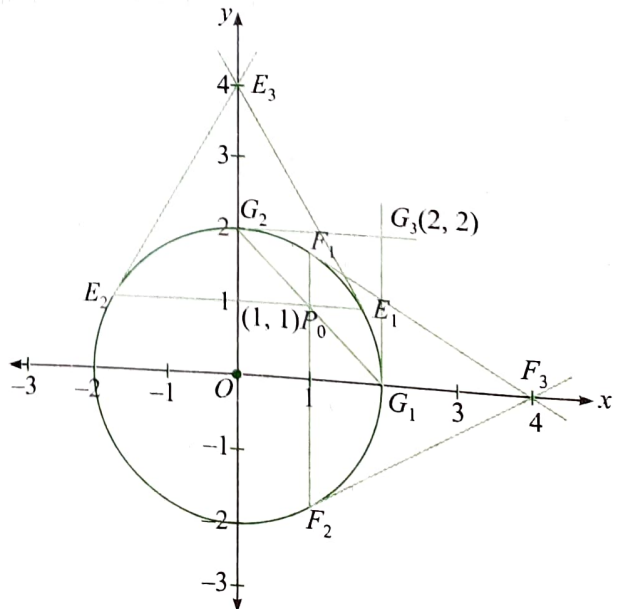
$$\text{or } 9m^2 = 1 + m^2$$

$$\text{or } m = \pm \frac{1}{2\sqrt{2}}$$

So, the equations of common tangents are

$$y = \frac{1}{2\sqrt{2}}(x - 6), y = -\frac{1}{2\sqrt{2}}(x - 6), \text{ and also } x = 2$$

3. (1)



Circle is $x^2 + y^2 = 4$.

Equation of chord through $P_0(1, 1)$ parallel to x -axis is $y = 1$.

Solving this with circle, we get $E_1(\sqrt{3}, 1)$ and $E_2(-\sqrt{3}, 1)$.

Equations of tangent to circle at E_1 and E_2 are $\sqrt{3}x + y = 4$ and $-\sqrt{3}x + y = 4$, respectively.

These tangents meet at $E_3(0, 4)$.

Chord through $P_0(1, 1)$, parallel to y -axis is $x = 1$, which meets circle at $F_1(1, \sqrt{3})$ and $F_2(1, -\sqrt{3})$.

Tangents to circle at F_1 and F_2 meet at $F_3(0, 4)$.

Chord through P_0 having slope -1 is $x + y = 2$ which meets circle at $G_1(0, 2)$ and $G_2(2, 0)$.

Tangents to circle at G_1 and G_2 meet at $G_3(2, 2)$.

Clearly, E_3, F_3 and G_3 lie on line $x + y = 4$.

4. (4) Let the coordinates of P be $(2 \cos \theta, 2 \sin \theta)$.

Equation of tangent to circle at P is $x \cos \theta + y \sin \theta = 2$

$$\therefore M\left(\frac{2}{\cos \theta}, 0\right), N\left(0, \frac{2}{\sin \theta}\right)$$

Let the mid-point of MN be (x, y) .

$$\therefore x = \frac{1}{\cos \theta} \text{ and } y = \frac{1}{\sin \theta}$$

Squaring and adding, we get

$$\frac{1}{x^2} + \frac{1}{y^2} = 1$$

or $x^2 + y^2 = x^2 y^2$; this is the required locus.

Matrix Match Type

1. $a \rightarrow p$

$$\frac{|0+0-1|}{\sqrt{h^2+k^2}} = 2$$

$$\Rightarrow 1 = 4(k^2 + h^2)$$

$$\therefore h^2 + k^2 = \left(\frac{1}{2}\right)^2 \text{ which is a circle.}$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (8) P is mid-point of AB and $DP \perp AB$.

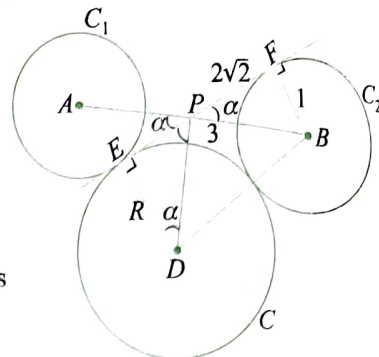
$$\text{In } \triangle PFB \cos \alpha = \frac{2\sqrt{2}}{3}$$

$$\therefore \sin \alpha = \frac{1}{3}$$

In $\triangle DEP$

$$\tan \alpha = \frac{2\sqrt{2}}{R}$$

$$\text{or } R = \frac{2\sqrt{2}}{\tan \alpha} = 8 \text{ units}$$

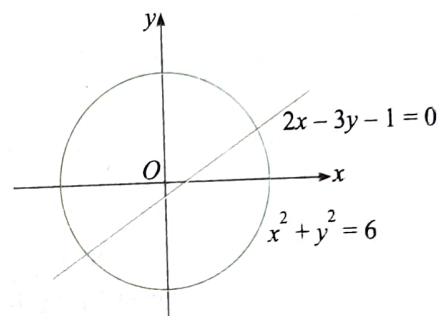


2. (2) $L: 2x - 3y - 1 = 0$

$$S: x^2 + y^2 - 6 = 0$$

If $L_1 > 0$ and $S_1 < 0$, then the point lies in the smaller part.

Therefore, $(2, 3/4)$ and $(1/4, -1/4)$ lie inside



3. (2) **Case I:** When $p = 0$ (i.e., circle passes through origin).

Here, equation of circle is $x^2 + y^2 + 2x + 4y = 0$.

Clearly, this circle meets coordinate axes at exactly three points.

Case II: When circle intersects x -axis at 2 distinct points and touches y -axis.

This occurs when $(g^2 - c) > 0$ and $f^2 - c = 0$

$$\Rightarrow 1 - (-p) > 0 \text{ and } 4 - (-p) = 0$$

$$\Rightarrow p > -1 \text{ and } p = -4$$

Not possible.

Case III: When circle intersects y -axis at 2 distinct points and touches x -axis.

This occurs when $g^2 - c = 0$ and $(f^2 - c) > 0$

$$\Rightarrow 1 - (-p) = 0 \text{ and } 4 - (-p) > 0$$

$$\Rightarrow p = -1 \text{ and } p > -4$$

Here, $p = -1$ is possible.

Hence, possible values of p are 0 and -1 .

Chapter 5

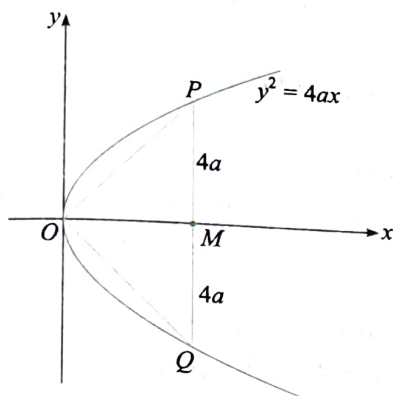
Concept Application Exercises

Exercise 5.1

1. From the figure, y coordinate of point P is $4a$.

So, from the equation of parabola

$$(4a)^2 = 4ax \text{ or } x = 4a$$



Therefore, coordinates of point P are $(4a, 4a)$.

$$\therefore \angle POM = 45^\circ$$

$$\therefore \angle POQ = 90^\circ$$

2. Focal distance of point P is $(a + x)$ or $(a + 5)$.

Given that $a + 5 = 10$

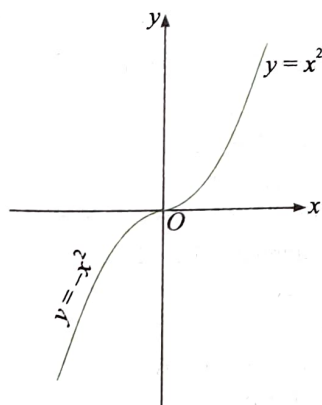
$$\therefore a = 5$$

3. (a) $y = x|x|$

$$\therefore y = \begin{cases} x \cdot (-x), & x < 0 \\ x \cdot x, & x \geq 0 \end{cases}$$

$$\text{or } y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

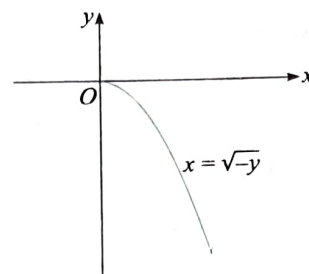
So, equation represents parts of two parabolas as shown in the figure.



(b) $x = \sqrt{-y}, y \leq 0$

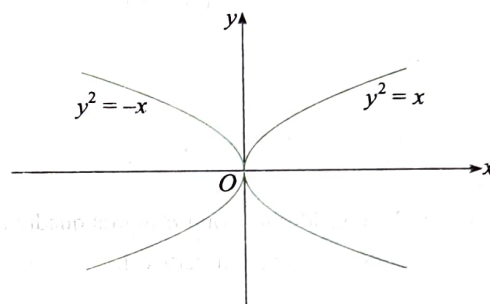
$$\therefore x^2 = -y, y \leq 0$$

So, equation represents part of the parabola lying in fourth quadrant.



(c) $x^2 = y^4$
 $\therefore y^2 = \pm x$

So, equation represents two parabolas.

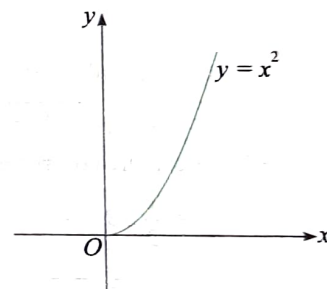


(d) $x = e^t, 2t = \log_e y$

$$\therefore x = e^t, y = e^{2t}$$

$$\therefore y = x^2, x > 0$$

So, equation represents part of the parabola lying in first quadrant.



4. The parabolas are $y^2 - x = 0$ and $y^2 + x = 0$. The point $(\lambda, -1)$ is an exterior point if

$$1 - \lambda > 0 \text{ and } 1 + \lambda > 0$$

$$\text{or } \lambda < 1 \text{ and } \lambda > -1$$

$$\text{or } -1 < \lambda < 1$$

5. Let the point P be $(at^2, 2at)$ and focus S be $(a, 0)$.

$$\text{Now, } SP = at^2 + a = k$$

(given) (1)

Let (α, β) be the moving point. Then

$$\alpha = at^2 \text{ and } \beta = 2at$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{t}{2} \text{ and } a = \frac{\beta^2}{4\alpha} \quad (\because \text{Point } (\alpha, \beta) \text{ lies on } y^2 = 4ax)$$

On substituting these values in equation (1), we get

$$\frac{\beta^2}{4\alpha} \left(1 + \frac{4\alpha^2}{\beta^2} \right) = k$$

$$\Rightarrow \beta^2 + 4\alpha^2 = 4k\alpha$$

$$\Rightarrow 4x^2 + y^2 - 4kx = 0;$$

which is the required locus.

6. Let the midpoint of chord of the parabola be $P(h, k)$.

So, equation of chord is

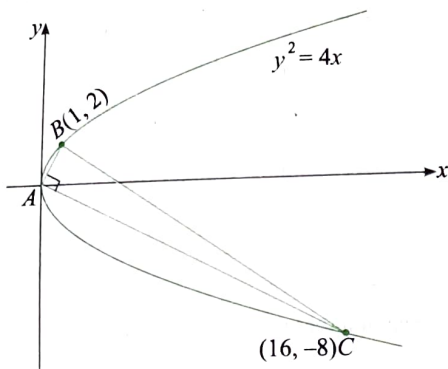
$$ky - 2a(x + h) = k^2 - 4ah$$

This chord passes through the point $(3a, a)$.

$$\therefore ak - 2a(3a + h) = k^2 - 4ah$$

So, locus of point P is $y^2 - 2ax - ay + 6a^2 = 0$.

7. Let point B have coordinates $(t^2, 2t)$.



$$AB = \sqrt{5}$$

$$\therefore t^4 + 4t^2 = 5$$

$$\Rightarrow (t^2 - 1)(t^2 + 5) = 0$$

$$\Rightarrow t = \pm 1$$

$\therefore B \equiv (1, 2)$ (considering point B in first quadrant)

Now, BC subtends right angle at vertex A .

$$\therefore t t' = -4$$

$$\therefore t' = -4$$

So, C has coordinates $(16, -8)$.

$$\therefore AC = \sqrt{256 + 64} = \sqrt{320}$$

Therefore, area of triangle ABC

$$= \frac{1}{2} AB \times AC$$

$$= \frac{1}{2} \sqrt{5} \sqrt{320}$$

$$= \frac{1}{2} \sqrt{1600} = 20 \text{ sq. units}$$

8. $P \equiv (t_1^2, 2t_1)$ and $Q \equiv (t_2^2, 2t_2)$

L is the midpoint of PQ .

$$\therefore L \equiv \left(\frac{1}{2} (t_1^2 + t_2^2), (t_1 + t_2) \right)$$

$$\text{Slope of } PQ = \frac{2}{t_1 + t_2}$$

Equation of perpendicular bisector of chord PQ is

$$y - (t_1 + t_2) = \frac{-(t_1 + t_2)}{2} \left(x - \frac{t_1^2 + t_2^2}{2} \right)$$

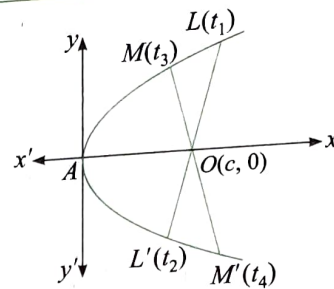
Putting $y = 0$, we get

$$M \equiv \left(2 + \frac{t_1^2 + t_2^2}{2}, 0 \right)$$

$$N \equiv \left(\frac{t_1^2 + t_2^2}{2}, 0 \right)$$

$$\therefore MN = OM - ON = 2$$

9.



$$\text{Equation of } LOL': 2x - (t_1 + t_2)y + 2at_1t_2 = 0$$

Since chord LOL' passes through $(c, 0)$, we have

$$t_1t_2 = -\frac{c}{a}$$

Similarly, for chord MOM' ,

$$t_3t_4 = -\frac{c}{a}$$

Now, the equation of the circle with LL' as diameter is

$$(x - at_1^2)(x - at_2^2) + (y - 2at_1)(y - 2at_2) = 0$$

$$\text{or } x^2 + y^2 - a(t_1^2 + t_2^2)x - 2a(t_1 + t_2)y + c^2 - 4ac = 0 \quad (1)$$

(As $a^2t_1^2t_2^2 = c^2$ and $4a^2t_1t_2 = -4ac$)

Similarly, for the circle with MM' as diameter, the equation is

$$x^2 + y^2 - a(t_3^2 + t_4^2)x - 2a(t_3 + t_4)y + c^2 - 4ac = 0 \quad (2)$$

Radical axis of (1) and (2) is (1) - (2)

$$\text{or } a(t_1^2 + t_2^2 - t_3^2 - t_4^2)x - 2a(t_1 + t_2 - t_3 - t_4)y = 0$$

which passes through the origin (vertex of the parabola).

Exercise 5.2

1. Focus is $(0, 2)$ and vertex is $(0, 4)$.

So, axis of the parabola is y -axis.

Also, parabola is concave downward.

Distance between focus and directrix is $a = 2$.

$$\therefore \text{Latus rectum} = 4a = 8$$

So, using equation $(x - h)^2 = 4a(y - k)$, equation of parabola is

$$(x - 0)^2 = -8(y - 4)$$

$$\text{or } x^2 + 8y = 32$$

2. From the definition of parabola, any point on the parabola $P(x, y)$ is equidistant from focus and directrix.

$$\text{So, } (x - 0)^2 + (y - 1)^2 = \left(\frac{x + 2}{\sqrt{1}} \right)^2$$

$$\text{or } (y - 1)^2 = 4(x + 1)$$

Thus, vertex of parabola is $(-1, 1)$.

3. We have $x^2 = 2(2x + y)$

$$\therefore x^2 - 4x + 4 = 2(y + 2)$$

$$\Rightarrow (x - 2)^2 = 2(y + 2)$$

Comparing with $(x - h)^2 = 2(y - k)$, we get

Vertex of the parabola is $(2, -2)$.

Length of latus rectum $= 4a = 2$

Axis of the parabola is $x = 2$.

Also, parabola is concave upward.

Focus lies at distance a units above the vertex on the axis.

$$\text{So, focus is } \left(2, -2 + \frac{1}{2} \right) \equiv \left(2, -\frac{3}{2} \right).$$

Directrix is at distance a units below the vertex and perpendicular to the axis.

$$\text{Therefore, directrix is } y = -2 - \frac{1}{2} \text{ or } y = -\frac{5}{2}.$$

4. Focus is midpoint of the extremities of latus rectum.
Thus, focus is $(2, 0)$.

Distance between focus and vertex is $a = 2$.

Also, axis of the parabola is $x = 2$ and parabola is concave downward as focus lies below the vertex.

Therefore, using equation $(x - h)^2 = -4a(y - k)$, required equation of parabola is

$$(x - 2)^2 = -8(y - 2)$$

5. Line joining $(1, 2)$ and $(2, 1)$ and line joining $(3, 4)$ and $(4, 3)$ are parallel.

Also, the perpendicular bisector of any two parallel chords is axis of parabola.

So, axis of the parabola is $x - y = 0$.

6. Solving given equations, we have

$$-x^2 - 4x = 4x + 12$$

$$\Rightarrow x^2 + 8x + 12 = 0$$

$$\Rightarrow x = -2, (x = -6 \text{ is not possible})$$

From equation of parabola,

$$y^2 = 4(-2 + 3) = 4$$

$$\therefore y = \pm 2.$$

So, curves intersect at $(-2, 2)$ and $(-2, -2)$.

So, length of common chord is 4.

7. Since the equation of latus rectum and the equation of tangent both are parallel, and the distance between them is one-fourth of the latus rectum, the distance between them is

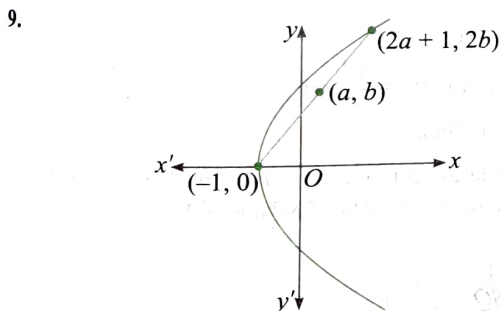
$$a = \left| \frac{-8 + 12}{\sqrt{1^2 + 1^2}} \right| = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\therefore \text{Length of latus rectum} = 4a = 4(2\sqrt{2}) = 8\sqrt{2}$$

8. Perpendicular distance of the focus from the directrix

$$2a = \left| -\frac{7}{\sqrt{17}} \right| = \frac{7}{\sqrt{17}}$$

$$\therefore \text{Length of latus rectum} = 4a = \frac{14}{\sqrt{17}}$$



Since $((2a + 1), 2b)$ lies on $y^2 = 4(x + 1)$, we have

$$4b^2 = 4(2a + 2)$$

$$\text{or } b^2 = 2(a + 1)$$

10. Slope of the chord joining point $P(t_1)$ and $Q(t_2)$ on the parabola $y^2 = 4x$ is

$$\frac{2}{t_2 + t_1} = 2 \quad (\text{given})$$

$$t_2 + t_1 = 1$$

$\therefore t_2 + t_1 = 1$
Let point $R(h, k)$ divide PQ in the ratio $1 : 2$.

$$\text{So, } h = \frac{t_2^2 + 2t_1^2}{1 + 2} \text{ and } k = \frac{2t_2 + 2(2t_1)}{1 + 2}$$

$$\Rightarrow 3h = t_2^2 + 2t_1^2$$

$$\text{and } t_2 + 2t_1 = (3k)/2$$

From (1) and (3), we get

$$t_1 = \frac{3}{2}k - 1 \text{ and } t_2 = 2 - \frac{3}{2}k$$

Putting these values in (2), we get

$$\left(2 - \frac{3}{2}k\right)^2 + 2\left(\frac{3}{2}k - 1\right)^2 = 3h$$

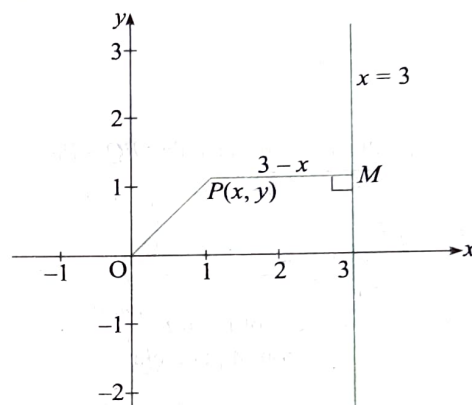
Hence, locus of the point R is

$$\left(y - \frac{8}{9}\right)^2 = \frac{4}{9}\left(x - \frac{2}{9}\right),$$

which is parabola having vertex at $\left(\frac{2}{9}, \frac{8}{9}\right)$.

11. Let variable point $P(x, y)$ lie in 1st quadrant.

$$\therefore x, y > 0$$



Also, point $P(x, y)$ is nearer to origin than to the line $x = 3$.

$$\text{Now, } OP = \sqrt{x^2 + y^2}$$

Distance of P from line $x - 3 = 0$ is $PM = 3 - x$.

According to the question,

$$OP < PM$$

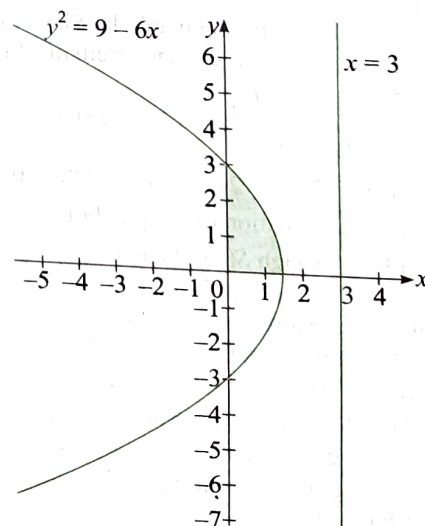
$$\Rightarrow \sqrt{x^2 + y^2} < (3 - x)$$

$$\Rightarrow x^2 + y^2 < x^2 - 6x + 9$$

$$\Rightarrow y^2 < -6x + 9$$

$$\Rightarrow y^2 < -6\left(x - \frac{3}{2}\right)$$

Points satisfying above inequality lie inside parabola $y^2 = 9 - 6x$ in first quadrant.

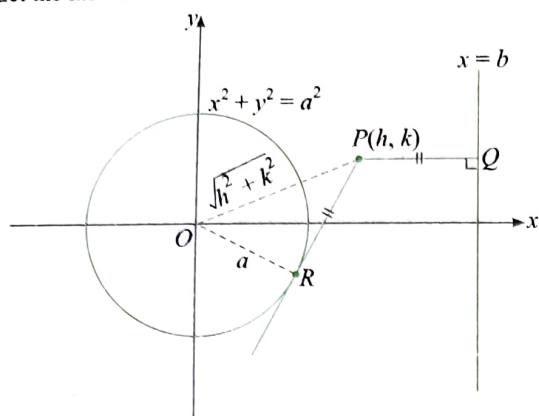


Parabola $y^2 = -6\left(x - \frac{3}{2}\right)$ is concave to the left, having axis as

x -axis, vertex at $\left(\frac{3}{2}, 0\right)$ and the directrix $x = 3$.

The required region is shown in the figure.

12. Let the fixed circle be $x^2 + y^2 = a^2$ and the fixed line be $y = b$.



Let variable point $P(h, k)$ be such that $PQ = PR$.

$$\therefore b - h = \sqrt{h^2 + k^2} - a$$

Squaring, we get

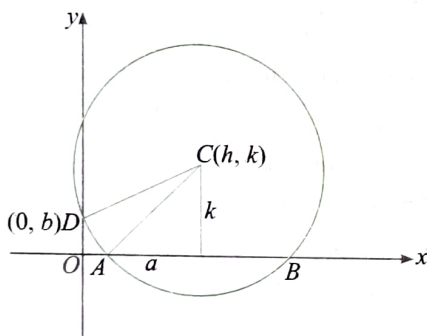
$$(b - h)^2 = h^2 + k^2 - a^2$$

$$\therefore b^2 - 2bh = y^2 - a^2,$$

which is equation of locus of point P .

Clearly, this is the equation of parabola.

13.



From the figure $AC = CD$

$$\therefore \sqrt{a^2 + k^2} = \sqrt{h^2 + (k - b)^2}$$

Squaring, we get

$$a^2 + k^2 = h^2 + k^2 - 2bk + b^2$$

So, equation of locus of point C is $x^2 - 2by + b^2 = a^2$.

Clearly, this is the equation of parabola.

14. Let $P(x, y)$ be any point on the parabola. Since the distance of P from the focus is equal to its distance from the directrix, we have

$$PS = PQ \text{ or } PS^2 = PQ^2$$

$$\text{or } (x + 1)^2 + (y - 1)^2 = \left[\frac{(4x + 3y - 24)}{5} \right]^2$$

$$\text{i.e., } 9x^2 + 16y^2 - 24xy + 242x + 94y - 526 = 0$$

This is the required equation of the parabola.

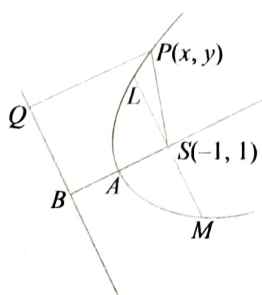
The axis is a line through $S(-1, 1)$ and perpendicular to the directrix $4x + 3y - 24 = 0$. Thus, the equation of the axis is

$$3(x + 1) - 4(y - 1) = 0$$

$$\text{or } 3x - 4y + 7 = 0$$

The axis and the directrix intersect at B . Solving them, we get $B(3, 4)$.

The vertex A is the midpoint of $S(-1, 1)$ and $B(3, 4)$.



Thus, $A \equiv \left(1, \frac{5}{2}\right)$

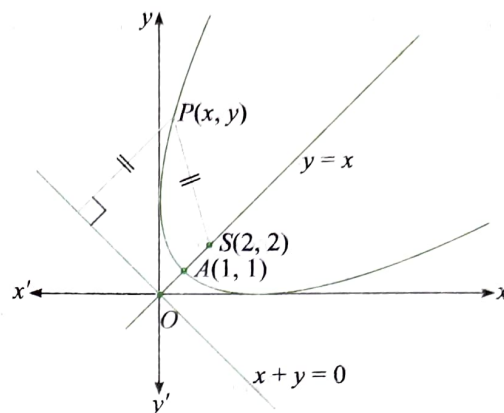
$$\text{Also, } AS = \sqrt{2^2 + \left(\frac{3}{2}\right)^2} = \frac{5}{2}$$

Hence, the length of the latus rectum is $4AS = 10$

Now, the latus rectum is a straight line through the focus S and parallel to the directrix.

Hence, its equation is $4x + 3y + 1 = 0$.

15. Axis of parabola is $y = x$.



Since vertex is at distance $\sqrt{2}$ from $(0, 0)$, vertex is $A(1, 1)$.

Also, focus is at distance $2\sqrt{2}$ from $(0, 0)$. So, focus is $S(2, 2)$.

Distance $SA = \sqrt{2}$

So, directrix is at distance $\sqrt{2}$ from origin, which is $x + y = 0$.

Hence, equation of the parabola is

$$\sqrt{(x - 2)^2 + (y - 2)^2} = \frac{|x + y|}{\sqrt{2}}$$

$$\text{or } x^2 + y^2 - 2xy = 8(x + y - 2)$$

$$\text{or } (x - y)^2 = 8(x + y - 2)$$

Exercise 5.3

1. We have $t_1 t_2 = -1$

$$\therefore \text{Product of roots} = \frac{t_2}{t_1} < 0$$

Hence, roots are real.

2. The length of latus rectum of $y = ax^2 + bx + c$ is $1/|a|$. Now, the semi-latus rectum is the HM of SP and SQ .

Then, we have

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{1/|2a|}$$

$$\text{or } 4|a| = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$\text{or } a = \pm \frac{5}{48}$$

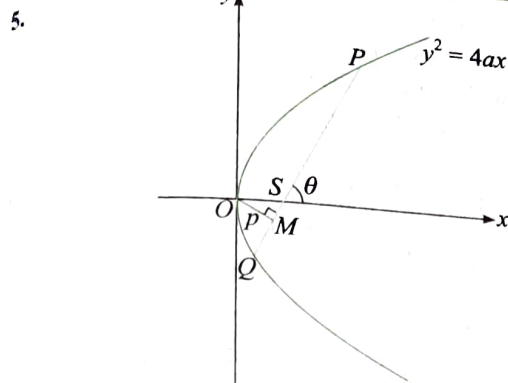
3. The length of focal chord making an angle α with the x -axis is $4a \operatorname{cosec}^2 \alpha$. For $\alpha \in [\pi/4, \pi/2]$, its maximum length is $4a \times 2 = 8a$ units.

4. The length of focal chord inclined at an angle θ with the positive x -axis is $4a \operatorname{cosec}^2 \theta$.

Given the length of focal chord is l . Then,

$$4a \operatorname{cosec}^2 \theta = l$$

$$\therefore \theta = \pm \sin^{-1} \sqrt{\frac{4a}{l}}$$



From the figure, $PQ = 4a \operatorname{cosec}^2 \theta = l$ (given)

Also, in triangle OMS , $\sin \theta = \frac{p}{a}$

$$\therefore l = 4a \frac{a^2}{p^2} \propto \frac{1}{p^2}$$

6. $x^2 - 4x + 6y + 10 = 0$

or $x^2 - 4x + 4 = -6 - 6y$

or $(x - 2)^2 = -6(y + 1)$

The circle drawn on focal distance as diameter always touches the tangent drawn to the parabola at vertex.

Thus, the circle will touch the line $y + 1 = 0$.

7. The extremities of the latus rectum are $(2, 4)$ and $(2, -4)$.

Since any circle drawn with any focal chord as its diameter touches the directrix, the radius of the circle is $2a = 4$ (since $a = 2$).

Exercise 5.4

1. The equation of any tangent to the parabola $y^2 = 4ax$ in terms of its slope m is

$$y = mx + \frac{a}{m}$$

The coordinates of the point of contact are $(a/m^2, 2a/m)$.

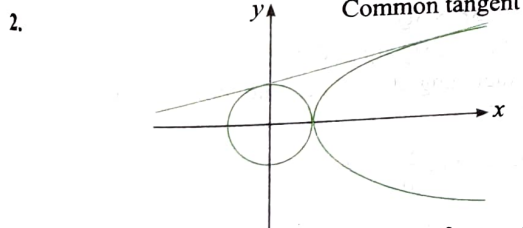
Therefore, the equation of tangent to $y^2 = ax$ is

$$y = mx + \frac{a}{4m}$$

and the coordinates of the point of contact are $(a/4m^2, a/2m)$.

It is given that $m = \tan 45^\circ = 1$.

Therefore, the coordinates of the point of contact are $(a/4, a/2)$.



The equation of tangent to the parabola $y^2 = 8ax$ having slope m is

$$y = mx + \frac{2a}{m} \quad (1)$$

The equation of tangent to the circle $x^2 + y^2 = 2a^2$ having slope m is

$$y = mx \pm \sqrt{2a^2(1 + m^2)} \quad (2)$$

Equations (1) and (2) are identical. Therefore,

$$\frac{2a}{m} = \pm \sqrt{2a^2(1 + m^2)}$$

or $m^4 + m^2 - 2 = 0$

or $(m^2 + 2)(m^2 - 1) = 0$

or $m^2 - 1 = 0$

or $m = \pm 1$

Hence, the required tangents are $x \pm y + 2a = 0$.

3. The given parabolas intersect at $(16, 8)$. The slope of the tangent to the parabola $y^2 = 4x$ at $(16, 8)$ is given by

$$m_1 = \left(\frac{dy}{dx} \right)_{(16, 8)} = \left(\frac{4}{2y} \right)_{(16, 8)} = \frac{2}{8} = \frac{1}{4}$$

The slope of the tangent to the parabola $x^2 = 32y$ at $(16, 8)$ is given by

$$m_2 = \left(\frac{dy}{dx} \right)_{(16, 8)} = \left(\frac{2x}{32} \right)_{(16, 8)} = 1$$

$$\therefore \tan \theta = \left| \frac{1 - \frac{1}{4}}{1 + \frac{1}{4}} \right| = \frac{3}{5}$$

or $\theta = \tan^{-1} \left(\frac{3}{5} \right)$

4. Line $\frac{1}{3}y = x + \frac{c}{3}$ touches the parabola $y^2 = 12x$.

Let us compare this line with $ty = x + at^2$ or $ty = x + 3t^2$.

So, we have $t = \frac{1}{3}$ which is parameter of point P .

Since PQ is focal chord, parameter of point Q is -3 .

Therefore, equation of tangent at Q is

$$(-3)y = x + 3(-3)^2$$

or $x + 3y + 27 = 0$

5. Eliminating y , we have

$$a - x = x - x^2 \text{ or } x^2 - 2x + a = 0$$

Since the line touches the parabola, we must have equal roots. Therefore,

$$4 - 4a = 0 \text{ or } a = 1$$

6. Any tangent to the parabola $y^2 = 8x$ is $y = mx + 2/m$, which is normal to the given circle.

Hence, the tangents must pass through the center $(-3, -4)$ of the circle.

Then, we have

$$-4 = -3m + \frac{2}{m}$$

or $3m^2 - 4m - 2 = 0$

or $m = \frac{4 \pm \sqrt{40}}{6} = \frac{2 \pm \sqrt{10}}{3}$

7. The line passing through the point $(0, 1)$ having slope m is

$$y - 1 = m(x - 0) \text{ or } y = mx + 1.$$

Solving this line with the given parabola, $9x^2 + 12x + 18y - 14 = 0$, we have

$$9x^2 + 12x + 18(mx + 1) - 14 = 0$$

or $9x^2 + 6(2 + 3m)x + 4 = 0$

Since the line touches the parabola, the above equation will have equal roots, i.e.,

$$D = 0$$

or $36(2 + 3m)^2 - 4(36) = 0$

or $9m^2 + 12m = 0$

i.e., $m = 0$ or $m = -\frac{4}{3}$

So, the equations of tangents are $4x + 3y - 3 = 0$ and $y - 1 = 0$.

8. Let the point be (h, k) . Let any tangent be

$$y = mx + \frac{a}{m}$$

or $k = mh + \frac{a}{m}$ or $m^2h - mk + a = 0$

Its roots are m_1 and $3m_1$. Therefore,

$$m_1 + 3m_1 = \frac{k}{h}$$

$$m_1 \cdot 3m_1 = \frac{a}{h}$$

Eliminating m_1 , we get the locus as

$$3y^2 = 16ax$$

9. The equation of tangent to the parabola having slope m is

$$y = mx + \frac{1}{m}$$

It passes through (h, k) . Therefore,

$$m^2h - mk + 1 = 0$$

$$\text{or } m_1 + m_2 = \frac{k}{h}, m_1m_2 = \frac{1}{h}$$

$$\text{Given } \theta_1 + \theta_2 = \frac{\pi}{4}$$

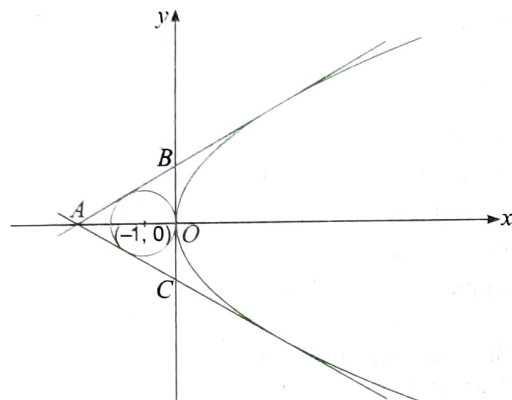
$$\text{or } \tan(\theta_1 + \theta_2) = 1$$

$$\text{or } \frac{m_1 + m_2}{1 - m_1m_2} = 1$$

$$\text{or } \frac{k}{h} = 1 - \frac{1}{h}$$

$$\text{or } y = x - 1$$

10.



We have parabola $y^2 = 4x$ and circle $(x+1)^2 + y^2 = 1$.

As shown in the figure, circle touches the parabola at origin.

So, one of the common tangents is y -axis.

The other two common tangents are symmetrical about x -axis.

Let equation of tangent AB to parabola be

$$y = mx + \frac{1}{m}$$

Solving it with the circle, we have

$$x^2 + \left(mx + \frac{1}{m}\right)^2 + 2x = 0$$

$$\text{or } (1+m^2)x^2 + 4x + \frac{1}{m^2} = 0$$

Since line touches the circle, above equation has two equal roots.

$\therefore$ Discriminant, $D = 0$

$$\Rightarrow 16 - \frac{4}{m^2}(1+m^2) = 0$$

$$\Rightarrow 4m^2 = 1 + m^2$$

$$\Rightarrow m = \pm \frac{1}{\sqrt{3}}$$

$$\text{i.e., } \angle BAO = \angle OAC = \frac{\pi}{6}$$

Hence, the triangle is equilateral.

11. Let the point T be (h, k) .

Equation of chord of contact to parabola $y^2 = 4ax$ w.r.t. T is

$$ky = 2a(x+h)$$

It passes through the point $(-a, b)$.

$$\therefore bk = 2a(-a+h)$$

So, locus of point T is $by = 2a(x-a)$.

12. We have parabola $(y-1)^2 = 4(x-1)$.

Here, directrix is y -axis, i.e., $x = 0$.

Let point P on the parabola be $(t^2 + 1, 2t + 1)$.

So, equation of tangent to parabola at point P is

$$t(y-1) = (x-1) + t^2$$

$$\text{or } x - ty + (t^2 + t - 1) = 0$$

$$\text{It meets directrix at } Q\left(0, \frac{t^2 + t - 1}{t}\right).$$

Now, Q is midpoint of RP .

$$\therefore \frac{h+t^2+1}{2} = 0 \text{ and } \frac{k+2t+1}{2} = \frac{t^2+t-1}{t}$$

$$\Rightarrow t^2 = -1-h \text{ and } kt + 2t^2 + t = 2t^2 + 2t - 2$$

$$\Rightarrow t^2 = -1-h \text{ and } 2 = t(1-k)$$

Eliminating t , we get

$$4 = (-1-h)(1-k)^2$$

$$\text{or } (x+1)(y-1)^2 + 4 = 0,$$

which is the locus of point R .

13. The chord of contact of parabola $y^2 = 4x$ w.r.t. point $P(h, 2)$ is

$$2y = 2(x+h)$$

$$\text{or } x - y + h = 0$$

Distance of this chord from point P is 4.

$$\therefore \frac{|h-2+h|}{\sqrt{2}} = 4$$

$$\Rightarrow |h-1| = 2\sqrt{2}$$

$$\Rightarrow h = 1 \pm 2\sqrt{2}$$

$$\Rightarrow h = 1 - 2\sqrt{2} \text{ (as for } h = 1 + 2\sqrt{2} \text{ point lies inside parabola)}$$

Exercise 5.5

1. Tangents to parabola $y^2 = 4ax$ at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ meet at $T(at_1t_2, a(t_1+t_2))$.

$$\text{Now, } SP = a + at_1^2 \text{ and } SQ = a + at_2^2$$

$$\begin{aligned} \therefore (ST)^2 &= (at_1t_2 - a)^2 + (a(t_1+t_2))^2 \\ &= a^2(1+t_1^2)(1+t_2^2) \\ &= (a + at_1^2)(a + at_2^2) \\ &= SP \times SQ \end{aligned}$$

Hence, SP , ST and SQ are in G.P.

2. We know that tangents at the end point of focal chord are perpendicular.

$$\text{For given parabola, } \frac{dy}{dx} = 2x - 2$$

$$\therefore \left(\frac{dy}{dx}\right)_{(2,3)} = 2 = \text{slope of tangent at point } P$$

Thus, slope of tangent at point Q is $-\frac{1}{2}$.

3. We know that perpendicular tangents intersect on directrix.

So, there exists at least one point on the circle $x^2 + y^2 = a^2$ from which two perpendicular tangents can be drawn to parabola if circle touches or intersects the directrix.

Directrix of the parabola is $x = -1/2$.

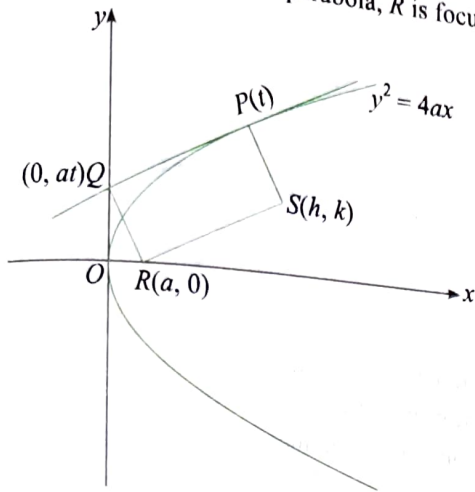
$$\therefore \text{Radius of the circle, } a \geq 1/2$$

4. The line $y = x - 1$ passes through $(1, 0)$. That means it is a focal chord. Hence, the required angle is 90° .

5. The origin $(0, 0)$ lies on the directrix of the given parabola which is $y = 0$. Then, the angle between the tangents is 90° .

6. We know that perpendicular tangents meet at the directrix.
 The given parabola is
 $y^2 + 4y - 6x - 2 = 0$
 or $(y + 2)^2 = 6(x + 1)$
 The equation of directrix is
 $x + 1 = -\frac{6}{4}$ or $x = -\frac{5}{2}$
 or $2x + 5 = 0$

7. From the property of tangent of parabola, R is focus.



From the figure, product of slopes of SR and PS is -1 .

$$\therefore \frac{2at - k}{at^2 - h} \times \frac{k - 0}{h - a} = -1 \quad (1)$$

Also,

Slope of tangent at PQ = Slope of RS

$$\therefore \frac{k}{h - a} = \frac{1}{t} \Rightarrow t = \frac{h - a}{k}$$

Putting the value of t into (1), we get

$$\left(2a \left(\frac{h - a}{k} \right) - k \right) k = (a - h) \left(a \left(\frac{h - a}{k} \right)^2 - h \right)$$

$$\Rightarrow 2a(x - a) - y^2 = (a - x) \left(a \left(\frac{x - a}{y} \right)^2 - x \right),$$

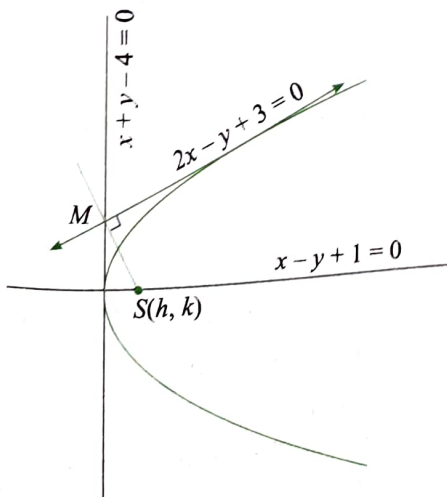
which is the required locus.

8. We have

$$\text{axis: } L \equiv x - y + 1 = 0$$

$$\text{Tangent at vertex: } T_1 \equiv x + y - 4 = 0$$

$$\text{Tangent at some point on the parabola: } T \equiv 2x - y + 3 = 0$$



Let the focus be $S(h, k)$.

Since focus lies on the axis L ,

$$h - k + 1 = 0$$

Now, foot of perpendicular from focus upon tangent T lies on tangent at vertex T_1 .

So, solving tangents T_1 and T , we get $M\left(\frac{1}{3}, \frac{11}{3}\right)$, which is required foot of perpendicular.

Now, slope of T is 2.

$$\therefore \text{Slope of } SM = \frac{k - \frac{11}{3}}{h - \frac{1}{3}} = -\frac{1}{2}$$

$$\text{or } 3h + 6k - 23 = 0 \quad (2)$$

Solving (1) and (2), we get

$$h = \frac{17}{9} \text{ and } k = \frac{26}{9}$$

$$\therefore \text{Focus} \equiv \left(\frac{17}{9}, \frac{26}{9} \right)$$

Exercise 5.6

1. Equation of normal to the parabola $y^2 = 4ax$ having slope m is
 $y = mx - 2am - am^3$

Comparing this equation with $y = \sqrt{2}x - 4a\sqrt{2}$, we get

$$m = \sqrt{2}$$

$$\text{and } 2am + am^3 = 2\sqrt{2}a + 2\sqrt{2}a = 4a\sqrt{2}$$

Thus, given line is normal to parabola.

The point on the parabola is $(am^2, -2am) \equiv (2a, -2\sqrt{2}a)$.

2. We have parabola

$$y = x^2 - 3x - 4$$

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = 2x - 3$$

$$(a) \left(\frac{dy}{dx} \right)_{(3, -4)} = 3$$

So, equation of normal is

$$y + 4 = 3(x - 3)$$

$$\text{or } 3x - y - 13 = 0$$

$$(b) \frac{dy}{dx} = 2x - 3 = 5$$

$$\therefore x = 4$$

$$\text{So, } y = 16 - 12 - 4 = 0.$$

Therefore, equation of normal is

$$y - 0 = 5(x - 4)$$

$$\text{or } 5x - y - 20 = 0$$

3. The equation of a normal to the parabola $y^2 = 24x$ having slope m is $y = mx - 12m - 6m^3$.

It is parallel to $y = 2x + 3$. Therefore, $m = 2$.

Then the equation of the parallel normal is

$$y = 2x - 24 - 48 = 2x - 72$$

The distance between $y = 2x + 3$ and $y = 2x - 72$ is

$$\left| \frac{72 + 3}{\sqrt{4 + 1}} \right| = \frac{75}{\sqrt{5}}$$

4. The slope of the given line is $m = \cos \theta$.

Also, normal to parabola $y^2 = 16x$ having slope m is

$$y = mx - 8m - 4m^3$$

$$\therefore y = x \cos \theta - 8 \cos \theta - 4 \cos^3 \theta$$

$$\therefore y = x \cos \theta - 11 \cos \theta + 3 \cos \theta - 4 \cos^3 \theta$$

$$\therefore y = (x - 11) \cos \theta - \cos 3\theta$$

So, this line is always normal to parabola $y^2 = 16x$.

(1)

5. Normal at $P(at^2, 2at)$ is $y = -tx + 2at + at^3$.
It meets the axis $y = 0$ at $G(2a + at^2, 0)$.

If (x, y) is the midpoint of PG , then

$$2x = 2a + at^2 + at^2, 2y = 2at$$

Eliminating t , we have

$$x - a = at^2 = a\left(\frac{y}{a}\right)^2$$

$$\text{or } y^2 = a(x - a)$$

which is the required locus.

6. For parabola $y^2 = 4ax$, equation of normal having slope m is

$$y = mx - 2am - am^3 \quad (1)$$

For parabola $y^2 = 4c(x - b)$, equation of normal having slope m is

$$y = m(x - b) - 2cm - cm^3 \quad (2)$$

For common normal to parabolas, equations (1) and (2) are identical.

$$\therefore -2am - am^3 = -bm - 2cm - cm^3$$

$$\Rightarrow (c - a)m^2 = 2(a - c) - b \quad (\text{as } m \neq 0)$$

$$\Rightarrow m^2 = \frac{2(a - c) - b}{c - a}$$

For real value of m ,

$$\frac{2(a - c) - b}{c - a} > 0$$

$$\Rightarrow -2 + \frac{b}{a - c} > 0$$

$$\Rightarrow \frac{b}{a - c} > 2$$

7. Equation of normal to parabola $y^2 = 4ax$ having slope m is

$$y = mx - 2am - am^3$$

For parabola $y^2 = x$, $a = \frac{1}{4}$.

Thus, equation of normal is

$$y = mx - \frac{m}{2} - \frac{m^3}{4}$$

This normal passes through $(c, 0)$.

$$\therefore mc - \frac{m}{2} - \frac{m^3}{4} = 0 \quad (1)$$

$$\Rightarrow m \left[c - \frac{1}{2} - \frac{m^2}{4} \right] = 0$$

$$\Rightarrow m = 0 \text{ or } m^2 = 4 \left(c - \frac{1}{2} \right) \quad (2)$$

For, $m = 0$ normal is $y = 0$, i.e., x -axis.

For other two normals to be real $m^2 \geq 0$

$$\Rightarrow 4 \left(c - \frac{1}{2} \right) \geq 0 \text{ or } c \geq \frac{1}{2}$$

For $c = \frac{1}{2}$, $m = 0$.

Therefore, for other real normals, $c > 1/2$.

Now, other two normals are perpendicular to each other.

$\therefore$ Product of their slope $= -1$

So, from (1),

$$4 \left(\frac{1}{2} - c \right) = -1$$

$$\Rightarrow c = \frac{3}{4}$$

Exercise 5.7

1. A normal at point t_1 cuts the parabola again at t_2 . Then,

$$t_2 = -t_1 - \frac{2}{t_1}$$

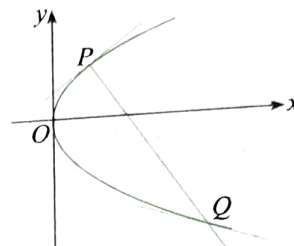
$$\text{or } t_1^2 + t_1 t_2 + 2 = 0$$

Since t_1 is real, discriminant is greater than 0. Therefore,

$$t_2^2 - 8 \geq 0$$

$$\text{or } t_2^2 \geq 8$$

2.



Slope of normal at point $P(t) = -t$

or Slope of line $PQ = -t$

Now, point Q has parameter

$$-t - \frac{2}{t}$$

$$\text{Slope of tangent at point } Q = \frac{1}{-t - \frac{2}{t}} = \frac{-t}{t^2 + 2}$$

Now, the angle between the normal and the parabola at Q is equivalent to the angle between the normal and the tangent at point Q . Therefore,

$$\tan \theta = \left| \frac{-t + \frac{t}{t^2 + 2}}{1 + t \frac{t}{t^2 + 2}} \right| = \left| \frac{t}{2} \right|$$

$$\text{Hence, } \theta = \tan^{-1} \left| \frac{t}{2} \right|$$

3. Let parameter of point Q be t' .

$$\therefore t' = -t - \frac{2}{t}$$

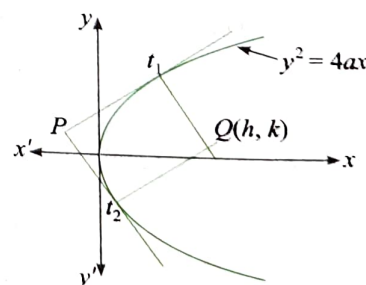
$$\Rightarrow t + t' = -\frac{2}{t}$$

$$\Rightarrow 2at + 2at' = -\frac{4a}{t} = 3$$

$$\Rightarrow 4a = -3t$$

(given)

4.



Let $P \equiv (at_1^2, a(t_1 + t_2))$; P must satisfy $y^2 = a(x + b)$. Hence,

$$a^2(t_1 + t_2)^2 = a\{(at_1^2) + b\}$$

$$\text{or } a(t_1^2 + t_2^2 + t_1 t_2) = b$$

Now, the coordinates of the point of intersection of normals at t_1 and t_2 are, respectively,

$$h = a(t_1^2 + t_2^2 + t_1 t_2 + 2) \quad (1)$$

$$\text{and } k = -at_1 t_2(t_1 + t_2) \quad (2)$$

From (1),

$$h = b + 2a$$

$$\text{or } x = b + 2a$$

5. Any point on the parabola $y^2 = 4x$, say $(t^2, 2t)$, lies on the line $x - 2y - 1 = 0$.

$$\therefore t^2 - 4t - 1 = 0$$

This equation has two roots t_1 and t_2 , which are parameters of the points of intersection of line and parabola.

$$\therefore t_1 + t_2 = 4 \text{ and } t_1 t_2 = -1$$

Now, point of intersection of normals at P and Q is

$$N \equiv (2 + (t_1^2 + t_2^2 + t_1 t_2), -t_1 t_2(t_1 + t_2))$$

$$\text{or } N \equiv (2 + (t_1 + t_2)^2 - t_1 t_2, -t_1 t_2(t_1 + t_2))$$

Putting the values of $t_1 + t_2$ and $t_1 t_2$, we get

$$N \equiv (2 + 16 + 1, 1(4)) \equiv (19, 4)$$

6. Point of intersection of normals to the parabola at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ is

$$R \equiv (2a + a(t_1^2 + t_2^2 + t_1 t_2), -at_1 t_2(t_1 + t_2)) \equiv (h, k)$$

Also, PQ is focal chord, then $t_1 t_2 = -1$.

$$\therefore k = a(t_1 + t_2)$$

$$\text{and } h = 2a + a[(t_1 + t_2)^2 + t_1 t_2]$$

$$\text{or } h = 2a + a[(t_1 + t_2)^2 + 1]$$

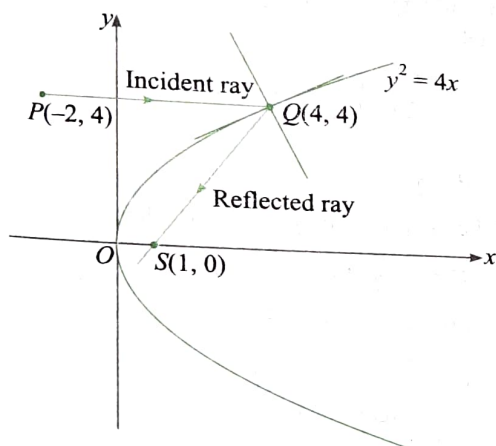
Eliminating $(t_1 + t_2)$ from (1) and (2), we get

$$h = 2a + a\left(\frac{k^2}{a^2} + 1\right)$$

$$\therefore y^2 = a(x - 3a),$$

which is the required locus.

7. Ray parallel to x -axis from point $P(-2, 4)$ strikes the parabola at $Q(4, 4)$.



So, incident ray SQ passes through the focus.

Equation of SQ is

$$y - 4 = \frac{4}{3}(x - 4)$$

$$\text{or } 4x - 3y - 4 = 0$$

8. Equation of normal to parabola $y^2 = 4ax$ having slope m is

$$y = mx - 2am - am^3$$

This normal passes through point $P(h, k)$

$$\therefore k = mh - 2am - am^3$$

$$\text{or } am^3 + (2a - h)m + k = 0$$

This equation has three real roots m_1, m_2 and m_3 ; which are slopes of three normals.

Given that $m_1 = \tan \theta_1, m_2 = \tan \theta_2$ and $m_3 = \tan \theta_3$

From equation (1), we have

$$m_1 + m_2 + m_3 = 0;$$

$$m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2a - h}{a};$$

$$\text{and } m_1 m_2 m_3 = \frac{-k}{a}.$$

$$\text{Now, } \theta_1 + \theta_2 + \theta_3 = \alpha$$

$$\therefore \tan(\theta_1 + \theta_2 + \theta_3) = \tan \alpha$$

$$\Rightarrow \frac{m_1 + m_2 + m_3 - m_1 m_2 m_3}{1 - m_1 m_2 - m_2 m_3 - m_3 m_1} = \tan \alpha$$

$$\Rightarrow \frac{0 + \left(\frac{k}{a}\right)}{1 - \left(\frac{2a - h}{a}\right)} = \tan \alpha$$

$$\Rightarrow y = \tan \alpha (x - a),$$

which is the required locus.

Exercises

Single Correct Answer Type

1. (2) $x = 3 \cos t, y = 4 \sin t$

Eliminating t , we have

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

which is an ellipse.

$$x^2 - 2 = 2 \cos t \text{ and } y = 4 \cos^2 \frac{t}{2}$$

$$\text{or } y = 2(1 + \cos t)$$

$$\text{and } y = 2\left(1 + \frac{x^2 - 2}{2}\right)$$

which is a parabola.

$$\sqrt{x} = \tan t; \sqrt{y} = \sec t$$

Eliminating t , we have

$$y - x = 1$$

which is a straight line.

$$x = \sqrt{1 - \sin t}$$

$$y = \sin \frac{t}{2} + \cos \frac{t}{2}$$

Eliminating t , we have

$$x^2 + y^2 = 1 - \sin t + 1 + \sin t = 2$$

which is a circle.

2. (4) Eliminating θ from the given equations, we get

$$y^2 = -4a(x - a)$$

which is a parabola but

$$0 \leq \cos^2 \theta \leq 1$$

$$\text{or } 0 \leq x \leq a$$

$$\text{and } -1 \leq \sin \theta \leq 1$$

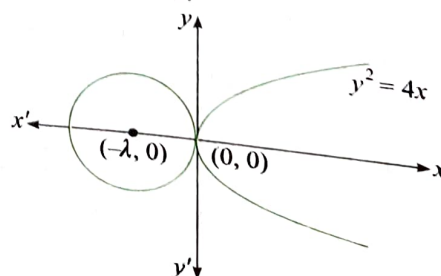
$$\text{or } -2a \leq y \leq 2a$$

Hence, the locus of point P is not whole parabola, but it is a part of the parabola.

3. (4) Any line passing through the focus other than the axis always meets the parabola at two distinct points.

Hence, $m \in \mathbb{R} - \{0\}$.

4. (1) The graph shows $\lambda > 0$.



5. (3) Let points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ lie on the parabola $y^2 = 4ax$.

Here, points P and Q are variable. But the slope of chord PQ ,

$$m_{PQ} = \frac{2}{t_1 + t_2}$$

Now, let the midpoint of PQ be $R(h, k)$. Then,

$$k = \frac{2at_1 + 2at_2}{2}$$

$$\text{or } k = a(t_1 + t_2) = \frac{2}{m}$$

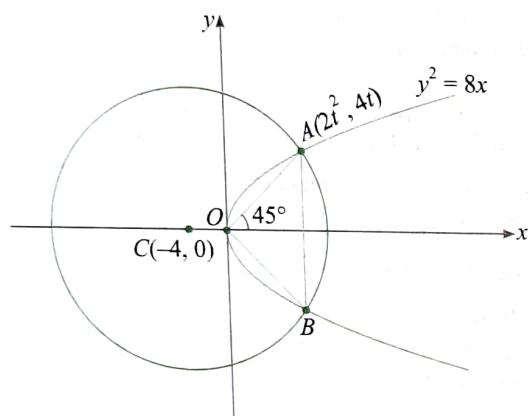
$$\therefore y = \frac{2}{m}$$

which is a line parallel to the axis of the parabola.

6. (1) Since points A and B are equidistant from focus, line AB is perpendicular to axis and so, parallel to directrix.

$$\therefore \text{Slope of directrix} = \text{Slope of } AB = \frac{5-3}{5-1} = \frac{1}{2}$$

7. (1) As shown in the figure, common chord AB subtends right angle at vertex.



$$\text{Slope of } OA = \frac{4t-0}{2t^2-0} = \tan 45^\circ$$

$$\Rightarrow \frac{2}{t} = 1 \Rightarrow t = 2$$

$$\therefore A \equiv (8, 8)$$

So, radius of circle = CA

$$= \sqrt{(8+4)^2 + (8-0)^2}$$

$$= \sqrt{144 + 64}$$

$$= 4\sqrt{13}$$

8. (3) Solving the circle $x^2 + y^2 = 5$ and the parabola $y^2 = 4x$, we have

$$x^2 + 4x - 5 = 0$$

i.e., $x = 1$ or $x = -5$ (not possible)

$$\therefore y^2 = 4 \text{ or } y = \pm 2$$

Therefore, the points of intersection are $P(1, 2)$ and $Q(1, -2)$.

Hence, $PQ = 4$.

9. (3) Let x_1, x_2 , and x_3 be the abscissae of the points on the parabola whose ordinates are y_1, y_2 , and y_3 , respectively. Then $y_1^2 = 4ax_1$, $y_2^2 = 4ax_2$, and $y_3^2 = 4ax_3$. Therefore, the area of the triangle whose vertices are (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) is

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} y_1^2/4a & y_1 & 1 \\ y_2^2/4a & y_2 & 1 \\ y_3^2/4a & y_3 & 1 \end{vmatrix} = \frac{1}{8a} \begin{vmatrix} y_1^2 & y_1 & 1 \\ y_2^2 & y_2 & 1 \\ y_3^2 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{8a} |(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$$

10. (2) Let $R(h, k)$ be the midpoint of PQ . Therefore, Q is $(2h-1, 2k)$. Since Q lies on $y^2 = 8x$, we get

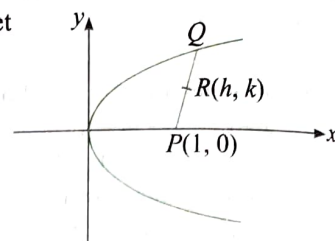
$$(2k)^2 = 8(2h-1)$$

Hence, the locus of $Q(h, k)$ is

$$y^2 = 2(2x-1)$$

$$\text{or } y^2 = 4x - 2$$

$$\text{or } y^2 - 4x + 2 = 0$$



11. (2) Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, -2at_2)$.

We have

$$m_{AS} = \tan\left(\frac{5\pi}{6}\right)$$

$$\text{or } \frac{2at_1}{at_1^2 - a} = -\frac{1}{\sqrt{3}}$$

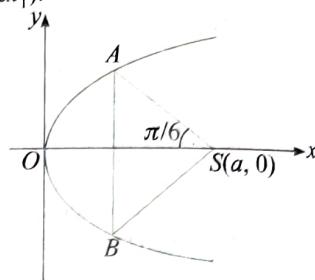
$$\text{or } t_1^2 + 2\sqrt{3}t_1 - 1 = 0$$

$$\text{or } t_1 = -\sqrt{3} \pm 2$$

Clearly, $t_1 = -\sqrt{3} - 2$ is rejected.

$$\text{Thus, } t_1 = (2 - \sqrt{3}).$$

$$\text{Hence, } AB = 4at_1 = 4a(2 - \sqrt{3}).$$



12. (4) Putting $x^2 = y/a$ in the circle $x^2 + (y-1)^2 = 1$, we get

$$\frac{y}{a} + y^2 - 2y = 0$$

(Note that for $a < 0$ they cannot intersect other than the origin)

Hence, we get

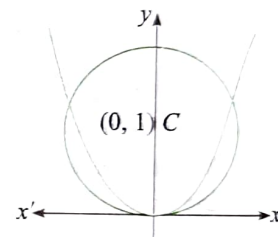
$$y = 0 \text{ or } y = 2 - \frac{1}{a}$$

Substituting $y = 2 - \frac{1}{a}$ in $y = ax^2$, we get

$$ax^2 = 2 - \frac{1}{a}$$

$$\text{or } x^2 = \frac{2a-1}{a^2} > 0$$

$$\text{or } a > \frac{1}{2}$$



13. (1) Let the midpoint of PQ be (α, β) . Then,

$$\alpha = x + \frac{c}{2} \text{ and } \beta = y + \frac{c}{2}$$

$$\text{or } \left(\beta - \frac{c}{2}\right)^2 = 4a\left(\alpha - \frac{c}{2}\right)$$

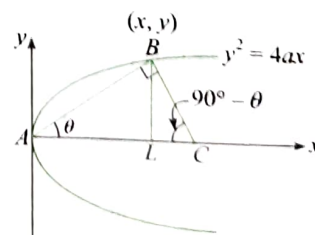
$$\text{or } \left(y - \frac{c}{2}\right)^2 = 4a\left(x - \frac{c}{2}\right)$$

which is the required locus.

14. (3) $\tan \theta = \frac{y}{x}$

The projection of BC on the x -axis,

$$LC = \frac{y}{\tan(90^\circ - \theta)} = y \tan \theta = \frac{y^2}{x} = 4a$$



15. (3) We must have

$$\alpha^2 + 1 - 4 < 0 \text{ and } 1 - 4\alpha < 0$$

$$\Rightarrow \alpha^2 < 3 \text{ and } \alpha > \frac{1}{4}$$

$$\Rightarrow -\sqrt{3} < \alpha < \sqrt{3} \text{ and } \alpha > \frac{1}{4}$$

$$\Rightarrow \frac{1}{4} < \alpha < \sqrt{3}$$

16. (2) Consider parabola $y^2 = 4ax$

So, $X \equiv (-a, 0)$

Let point P be $(at^2, 2at)$.

$$\therefore P' \equiv (at^2, -2at)$$

Equation of line PX is

$$y = \frac{2at - 0}{at^2 + a}(x + a)$$

$$\text{or } (1 + t^2)y = 2t(x + a)$$

Solving this line with given parabola, we get

$$\frac{4t^2(x + a)^2}{(1 + t^2)^2} = 4ax$$

$$\Rightarrow t^2(x + a)^2 = ax(1 + t^2)^2$$

$$\Rightarrow t^2(x^2 + a^2 + 2ax) = a(t^4 + 1 + 2t^2)x$$

$$\Rightarrow t^2x^2 + t^2a^2 = xat^4 + ax$$

$$\Rightarrow t^2x^2 - (a + at^4)x + a^2t^2 = 0$$

$$\Rightarrow xt^2(x - at^2) - a(x - at^2) = 0$$

$$\Rightarrow (x - at^2)(xt^2 - a) = 0$$

$$\Rightarrow x = at^2, a/t^2$$

Putting $x = a/t^2$ in (1), we get

$$(1 + t^2)y = 2t\left(\frac{a}{t^2} + a\right)$$

$$\therefore y = 2a/t$$

Thus, point Q is $(a/t^2, 2a/t)$.

Clearly, points Q and P' are extremities of focal chord.

So, $P'Q$ passes through the focus.

17. (3) The path of the water jet is a parabola.

Let its equation be

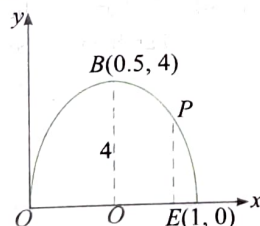
$$y = ax^2 + bx + c$$

It should pass through $(0, 0)$, $(0.5, 4)$, and $(1, 0)$. Therefore,

$$c = 0, a = -16, b = 16$$

$$\therefore y = -16x^2 + 16x$$

If $x = 0.75$, then $y = 3$.



18. (3) Consider equal parabola $y^2 = 12x$.

Vertex: $V(0, 0)$

Focus: $S(3, 0)$

End point of Latus Rectum: $A(3, 6)$

$$\therefore \text{Required area} = \frac{1}{2} SV \times AS = 9 \text{ sq. units}$$

19. (1) The family of parabolas is

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a = Ax^2 + Bx + C$$

and the vertex is $P(-B/2A, -D/4A) \equiv (h, k)$. Therefore,

$$h = -\frac{a^2/2}{2(a^3/3)} = -\frac{3}{4a}$$

$$\text{and } k = -\frac{(a^2/2)^2 - \{4a^3(-2a)/3\}}{4(a^3/3)}$$

$$\text{or } h = -\frac{3}{4a} \text{ and } k = -\frac{35a}{16}$$

Eliminating a , we have $hk = 105/64$.

Hence, the required locus is $xy = 105/64$.

20. (1) Let the focus be (a, b) .

The equations are

$$S_1: (x - a)^2 + (y - b)^2 = x^2$$

$$\text{and } S_2: (x - a)^2 + (y - b)^2 = y^2$$

Common chord $S_1 - S_2 = 0$ gives

$$x^2 - y^2 = 0$$

$$\text{or } y = \pm x$$

21. (3) $(\sqrt{3}h, \sqrt{3}k + 2)$ lies on the line $x - y - 1 = 0$. Therefore,

$$(\sqrt{3}h)^2 = (\sqrt{3}k + 2 + 1)^2$$

$$\text{or } 3h = 3k + 2 + 1 + 2\sqrt{3}k + 2$$

$$\text{or } 3^2(h - k - 1)^2 = 2^2(\sqrt{3}k + 2)^2$$

$$\text{or } 9(h^2 + k^2 + 1 - 2hk - 2h + 2k) = 4(3k + 2)$$

$$\text{or } 9(x^2 + y^2) - 18xy - 18x + 6y + 1 = 0$$

Now, $h^2 = ab$ and $\Delta \neq 0$.

Therefore, the locus is a parabola.

22. (3) Let $C_1(h, k)$ be the center of the circle.

The circle touches the x -axis. Then its radius is $r_1 = k$.

Also, the circle touches the circle with center $C_2(0, 3)$ and radius $r_2 = 2$. Therefore,

$$|C_1 C_2| = r_1 + r_2$$

$$\text{or } \sqrt{(h - 0)^2 + (k - 3)^2} = |k + 2|$$

Squaring, we get

$$h^2 - 10k + 5 = 0$$

Therefore, the locus is $x^2 - 10y + 5 = 0$, which is a parabola.

23. (4) Two parabolas are equal if the lengths of their latus rectums are equal.

The length of the latus rectum of $y^2 = \lambda x$ is λ .

The equation of the second parabola is

$$25\{(x - 3)^2 + (y + 2)^2\} = (3x - 4y - 2)^2$$

$$\text{or } \sqrt{(x - 3)^2 + (y + 2)^2} = \frac{|3x - 4y - 2|}{\sqrt{3^2 + 4^2}}$$

Here, the focus is $(3, -2)$ and the equation of the directrix is $3x - 4y - 2 = 0$.

$\therefore$ Length of latus rectum = $2 \times$ Distance between focus and directrix

$$= 2 \left| \frac{3 \times 3 - 4 \times (-2) - 2}{\sqrt{3^2 + (-4)^2}} \right| = 6$$

Thus, the two parabolas are equal if $\lambda = 6$.

24. (4) Length of latus rectum = $2 \times$ Distance of focus from directrix

$$= 2 \times \left| \frac{-\frac{u^2}{2g} \cos 2\alpha - \frac{u^2}{2g}}{\sqrt{1}} \right|$$

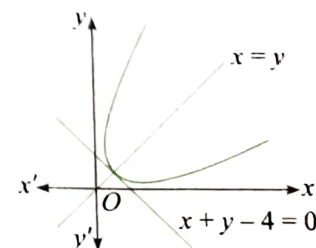
$$= \frac{2u^2}{g} \cos^2 \alpha$$

25. (1) $x^2 + y^2 - 2xy - 8x - 8y + 32 = 0$

$$\text{or } (x - y)^2 = 8(x + y - 4)$$

is a parabola whose axis is $x - y = 0$ and the tangent at the vertex is $x + y - 4 = 0$.

Also, when $y = 0$, we have $x^2 - 8x + 32 = 0$ which gives no real values of x .



When $x = 0$, we have $y^2 - 8y + 32 = 0$ which gives no real values of y .

So, the parabola does not intersect the axes. Hence, the graph falls in the first quadrant.

26. (1) $x = t^2 - t + 1, y = t^2 + t + 1$

or $x + y = 2(t^2 + 1)$ and $y - x = 2t$

or $\frac{x+y}{2} = 1 + \left(\frac{y-x}{2}\right)^2$

or $(y-x)^2 = 2(x+y) - 4$

or $(y-x)^2 = 2(x+y-2)$

Vertex will be the point where the lines $y-x=0$ and $x+y-2=0$ meet, i.e., the point $(1, 1)$.

27. (4) $y - \sqrt{3}x + 3 = 0$ can be rewritten as

$$\frac{y-0}{\sqrt{3}/2} = \frac{x-\sqrt{3}}{1/2} = r \quad (1)$$

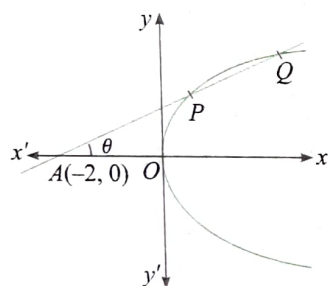
On solving (1) with the parabola $y^2 = x + 2$, we get

$$\frac{3r^2}{4} = \frac{r}{2} + \sqrt{3} + 2$$

or $3r^2 - 2r - (4\sqrt{3} + 8) = 0$

Hence, $AP \cdot AQ = |r_1 r_2| = \frac{4(\sqrt{3} + 2)}{3}$ (Product of roots)

28. (1) Let $(-2 + r \cos \theta, r \sin \theta)$ lie on the parabola. Then,



$$r^2 \sin^2 \theta - 4(-2 + r \cos \theta) = 0$$

or $r_1 + r_2 = \frac{4 \cos \theta}{\sin^2 \theta}, r_1 r_2 = \frac{8}{\sin^2 \theta}$

Now, $\frac{1}{AP} + \frac{1}{AQ} = \frac{r_1 + r_2}{r_1 r_2} = \frac{\cos \theta}{2}$

Given that $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$

or $\cos \theta < \frac{1}{2}$

or $\tan \theta > \sqrt{3} \quad [\because \cos \theta \text{ is decreasing and } \tan \theta \text{ is increasing in } (0, \pi/2)]$

$\therefore m > \sqrt{3}$

29. (4) Chord through $(2, 1)$ is

$$\frac{x-2}{\cos \theta} = \frac{y-1}{\sin \theta} = r \quad (1)$$

Solving (1) with the parabola $y^2 = x$, we have

$$(1 + r \sin \theta)^2 = 2 + r \cos \theta$$

or $\sin^2 \theta r^2 + (2 \sin \theta - \cos \theta)r - 1 = 0$

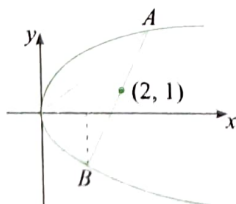
This equation has two roots: $r_1 = AC$ and $r_2 = -BC$. Then,

Sum of roots, $r_1 + r_2 = 0$

or $2 \sin \theta - \cos \theta = 0$ or $\tan \theta = \frac{1}{2}$

$$AB = |r_1 - r_2| = \sqrt{(r_1 + r_2)^2 - 4r_1 r_2}$$

$$= \sqrt{4 - \frac{1}{\sin^2 \theta}} = 2\sqrt{3}$$



30. (2) The joint equation of OA and OB is

$$x^2 - 4x(y-3x) - 4y(y-3x) + 20(y-3x)^2 = 0$$

(Making the equation of the parabola homogeneous using straight line)

or $193x^2 + 16y^2 - 112xy = 0$

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{56^2 - 193 \times 16}}{193 + 16}$$

$$= \frac{8\sqrt{3}}{209}$$

31. (3) $y^2 = 6\left(x - \frac{3}{2}\right)$

The equation of directrix is

$$x - \frac{3}{2} = -\frac{3}{2} \text{ or } x = 0$$

Vertex is $\left(\frac{3}{2}, 0\right)$ and focus is $S(3, 0)$.

Let the coordinates of P be $\left(\frac{3}{2} + \frac{3}{2}t^2, 3t\right)$.

Then the coordinates of M are $(0, 3t)$. Therefore,

$$MS = \sqrt{9 + 9t^2}$$

$$MP = \frac{3}{2} + \frac{3}{2}t^2$$

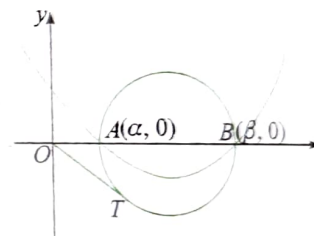
or $9 + 9t^2 = \left(\frac{3}{2} + \frac{3}{2}t^2\right)^2 = \frac{9}{4}(1 + t^2)^2$

or $4 = 1 + t^2$

$\therefore$ Length of side = 6

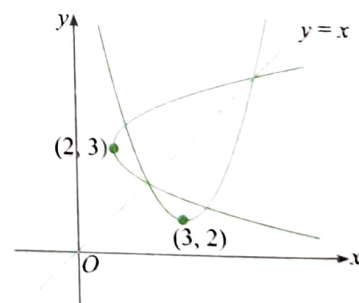
32. (4) $OT^2 = OA \cdot OB = \alpha\beta = \frac{c}{a}$

or $OT = \sqrt{\frac{c}{a}}$



33. (4) We have parabolas $x - 2 = (y - 3)^2$ and $y - 2 = (x - 3)^2$

Parabolas are symmetrical about line $y = x$.



These parabolas intersect at four points.

So, number of chords = ${}^4C_2 = 6$

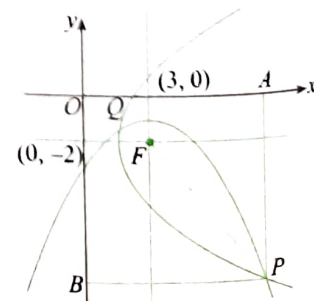
34. (1) Let the focus be F . The parabolas are open-down and open-right, respectively. Let the parabolas intersect at points P and Q . From P , perpendiculars are drawn on the x - and the y -axis at A and B , respectively. Then,

$$PA = PF = PB$$

So, P lies on the line $y = -x$.

Similarly, Q lies on the line $y = -x$. Therefore,

Slope of $PQ = -1$



35. (2) Let $P \equiv (at_1^2, 2at_1)$ and $Q \equiv (at_2^2, 2at_2)$.

$$\Rightarrow t_1 t_2 = -1$$

Equation of AP is $y = \frac{2}{t_1} x_1$.

$$\Rightarrow R \equiv (-a, -2a/t_1)$$

Similarly, $T \equiv (-a, -2a/t_2)$

$$\text{Slope of RS} = \frac{\left(\frac{-2a}{t_1}\right)}{2a} = -\frac{1}{t_2}$$

$$\text{Slope of TS} = \frac{\left(\frac{-2a}{t_2}\right)}{2a} = -\frac{1}{t_1}$$

$$\text{Now, } \left(-\frac{1}{t_1}\right) \times \left(-\frac{1}{t_2}\right) = \frac{1}{t_1 t_2} = -1$$

$$\Rightarrow \angle RST = 90^\circ$$

36. (3) Since the semi-latus rectum of a parabola is the harmonic mean between the segments of any focal chord of the parabola, SP , 4 , SQ are in HP. Therefore,

$$4 = \frac{2SP \times SQ}{SP + SQ}$$

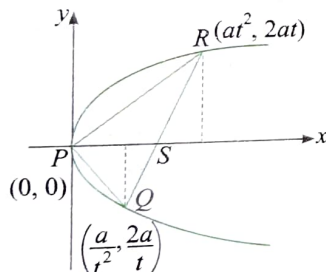
$$\text{or } 4 = \frac{2(6)(SQ)}{6 + SQ}$$

$$\text{or } 24 + 4(SQ) = 12(SQ)$$

$$\text{or } SQ = 3$$

37. (3) Difference of the ordinate

$$d = \left| 2at + \frac{2a}{t} \right| = 2a \left| t + \frac{1}{t} \right|$$



$$\text{Now, area } (A) = \frac{1}{2} \begin{vmatrix} at^2 & 2at & 1 \\ a/t^2 & 2a/t & 1 \\ 0 & 0 & 1 \end{vmatrix} = a^2 \left(t + \frac{1}{t} \right)$$

$$\text{or } 2a \left(t + \frac{1}{t} \right) = \frac{2A}{a}$$

38. (1) $A_1 B_1$ is a focal chord. Then $A_1 \equiv (at_1^2, 2at_1)$ and $B_1 \equiv (a/t_1^2, -2a/t_1)$.
 $A_2 B_2$ is a focal chord. Then $A_2 \equiv (at_2^2, 2at_2)$ and $B_2 \equiv (a/t_2^2, -2a/t_2)$.

Now, the equation of chord $A_1 A_2$ is

$$y(t_1 + t_2) - 2x - 2at_1 t_2 = 0 \quad (1)$$

Chord $B_1 B_2$ is

$$y \left(-\frac{1}{t_1} - \frac{1}{t_2} \right) - 2x - 2a \left(-\frac{1}{t_1} \right) \left(-\frac{1}{t_2} \right) = 0 \quad (2)$$

$$\text{or } y(t_1 + t_2) + 2xt_1 t_2 + 2a = 0$$

Subtracting (1) from (2), we get

$$2x(t_1 t_2 + 1) + 2a(t_1 t_2 + 1) = 0$$

$$\text{or } (x + a)(1 + t_1 t_2) = 0$$

$$\text{or } x + a = 0$$

Hence, they intersect on the directrix.

39. (3) The latus rectum of $y^2 = 2bx$ is $2b$.

The semi-latus rectum is b .

We know that semi-latus rectum is the HM of the segments of focal chord. Then,

$$\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$

$$\text{or } b = \frac{2ac}{a+c}$$

Now, for $ax^2 + bx + c = 0$,

$$D = b^2 - 4ac = \left(\frac{2ac}{a+c} \right)^2 - 4ac$$

$$= -4ac \left(\frac{a^2 + c^2 - ac}{(a+c)^2} \right) < 0$$

Hence, the roots are imaginary.

40. (2) The equation of tangent to the given parabola having slope m is

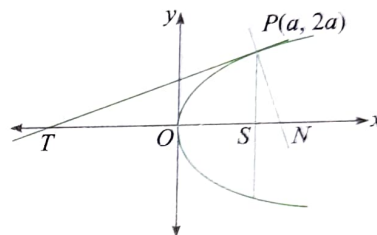
$$y = m(x + a) + \frac{a}{m}$$

$$\text{or } y = mx + am + \frac{a}{m}$$

Comparing (1) with $y = mx + c$, we have

$$c = am + \frac{a}{m}$$

41. (3)



One end of the latus rectum is $P(a, 2a)$.

The equation of tangent PT at $P(a, 2a)$ is

$$2ya = 2a(x + a) \text{ or } y = x + a$$

The equation of normal PN at $(a, 2a)$ is

$$y + x = 2a + a \text{ or } y + x = 3a$$

Solving $y = 0$ and $y = x + a$, we get

$$x = -a, y = 0$$

Solving $y = 0$ and $y + x = 3a$, we get

$$x = 3a, y = 0$$

The area of the triangle with vertices $P(a, 2a)$, $T(-a, 0)$, and $N(3a, 0)$ is $4a^2$.

42. (2) Let $P(x, y)$ be the point of contact. At P , both of them must have the same slope. We have

$$2y \frac{dy}{dx} = 4a, 2x = 4a \frac{dy}{dx}$$

Eliminating dy/dx , we get $xy = 4a^2$.

43. (1) The general equation of a parabola having its axis parallel to the y -axis is

$$y = ax^2 + bx + c$$

(1)

This is touched by the line $y = x$ at $x = 1$.

Therefore, the slope of the tangent at $(1, 1)$ is 1. Also, $x = ax^2 + bx + c$ must have equal roots, i.e.,

$$\left(\frac{dy}{dx} \right)_{(1,1)} = 1 \text{ and } (b-1)^2 = 4ac$$

$$\text{or } 2a + b = 1 \text{ and } (b-1)^2 = 4ac$$

Also, $(1, 1)$ lies on (1). Therefore,

$$a + b + c = 1$$

From $2a + b = 1$ and $a + b + c = 1$, we get

$$a - c = 0$$

$$\text{or } a = c$$

Substituting in $a + b + c = 1$, we get

$$2c + b = 1$$

$$\therefore 2f(0) + f'(0) = 1 \quad [\because f(0) = c \text{ and } f'(0) = b]$$

44. (4) Solving $y = 2x - 3$

$$\text{and } y^2 = 4a\left(x - \frac{1}{3}\right)$$

Eliminating y , we get

$$(2x - 3)^2 = 4a\left(x - \frac{1}{3}\right)$$

$$\text{or } 4x^2 + 9 - 12x = 4ax - \frac{4a}{3}$$

$$\text{or } 4x^2 - 4(3 + a)x + 9 + \frac{4a}{3} = 0$$

This equation must have equal roots, i.e.,

$$D = 0$$

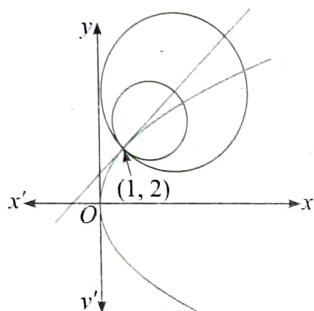
$$\text{or } 16(3 + a)^2 - 16\left(9 + \frac{4a}{3}\right) = 0$$

$$\text{or } 9 + a^2 + 6a = 9 + \frac{4a}{3}$$

$$\text{or } a^2 + \frac{14a}{3} = 0$$

$$\text{or } a = 0 \text{ or } a = -\frac{14}{3}$$

45. (1)



Tangent to the parabola $y^2 = 4x$ at $(1, 2)$ will be the locus, i.e.,

$$2y = 2(x + 1)$$

$$\text{or } y = x + 1$$

46. (2) $y = ax^2 - 6x + b$ passes through $(0, 2)$. Here,

$$2 = a(0^2) - 6(0) + b$$

$$\therefore b = 2$$

$$\text{Also, } \frac{dy}{dx} = 2ax - 6$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=3/2} = 2a\left(\frac{3}{2}\right) - 6$$

$$\text{or } 3a - 6 = 0$$

$$\therefore a = 2$$

47. (1) Let $A \equiv (at^2, 2at)$, $B \equiv (at^2, -2at)$. Then

$$m_{OA} = \frac{2}{t}, m_{OB} = \frac{-2}{t}$$

$$\text{Thus, } \left(\frac{2}{t}\right)\left(\frac{-2}{t}\right) = -1$$

$$\text{or } t^2 = 4$$

Thus, the tangents will intersect at $(-4a, 0)$.

48. (3) Let the tangents at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ make angles θ_1 and θ_2 with the x -axis.

$$\text{Then } \tan \theta_1 = \frac{1}{t_1} \text{ and } \tan \theta_2 = \frac{1}{t_2}.$$

Given that

$$\cos \theta_1 \cos \theta_2 = \lambda$$

$$\Rightarrow \sec^2 \theta_1 \sec^2 \theta_2 = \frac{1}{\lambda^2}$$

$$\Rightarrow (1 + \tan^2 \theta_1)(1 + \tan^2 \theta_2) = \frac{1}{\lambda^2}$$

$$\Rightarrow \left(1 + \frac{1}{t_1^2}\right)\left(1 + \frac{1}{t_2^2}\right) = \frac{1}{\lambda^2}$$

$$\Rightarrow \lambda^2[1 + (t_1 + t_2)^2 - 2t_1t_2 + t_1^2t_2^2] = t_1^2t_2^2$$

Now, point of intersection of tangents is

$$(at_1t_2, a(t_1 + t_2)) \equiv (h, k)$$

(Say)

$$\Rightarrow \lambda^2\left[1 + \frac{k^2}{a^2} - \frac{2h}{a} + \frac{h^2}{a^2}\right] = \frac{h^2}{a^2}$$

$$\Rightarrow x_2 = \lambda^2[(x - a)^2 + y^2]$$

49. (2) Tangent at point P is $ty = x + t^2$, where the slope of tangent is $\tan \theta = 1/t$.

Now, the required area is

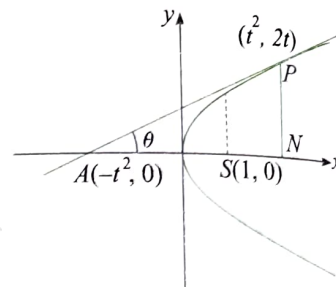
$$A = \frac{1}{2}(AN)(PN)$$

$$= \frac{1}{2}(2t^2)(2t)$$

$$= 2t^3 = 2(t^2)^{3/2}$$

Now, $t^2 \in [1, 4]$. Then $A_{\max}$ occurs when $t^2 = 4$.

Therefore, $A_{\max} = 16$.



50. (2) Tangent at point P is $ty = x + at^2$.

(1)

The line perpendicular to (1) passes through $(a, 0)$. Therefore,

$$y - 0 = -t(x - a)$$

$$\text{or } tx + y = ta$$

$$\text{or } y = t(a - x) \quad (2)$$

The equation of OP is

$$y - \frac{2}{t}x = 0 \text{ or } y = \frac{2}{t}x \quad (3)$$

From (2) and (3), eliminating t , we get

$$y^2 = 2x(a - x)$$

$$\text{or } 2x^2 + y^2 - 2ax = 0$$

51. (1) The slope of tangent at P is $1/t_1$ and that at Q is $1/t_2$. Therefore,

$$\cot \theta_1 = t_1 \text{ and } \cot \theta_2 = t_2$$

$$\text{Slope of } PQ = \frac{2}{t_1 + t_2}$$

$$\therefore \text{Slope of } OR = -\frac{t_1 + t_2}{2} = \tan \phi$$

(Angle in a semicircle is 90°)

$$\text{or } \tan \phi = -\frac{1}{2}(\cot \theta_1 + \cot \theta_2)$$

$$\therefore \cot \theta_1 + \cot \theta_2 = -2 \tan \phi$$

52. (3) Let $A \equiv (at_1^2, 2at_1)$

$$\text{and } B \equiv (at_1^2, -2at_1)$$

The equations of tangents at A and B are, respectively,

$$t_1y = x + at_1^2$$

$$\text{and } -t_1y = x + at_1^2$$

These tangents meet the y -axis, respectively, at

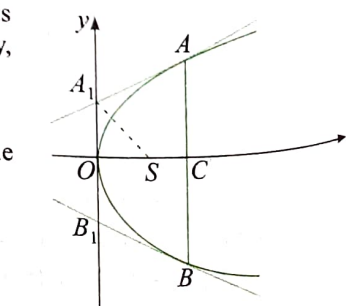
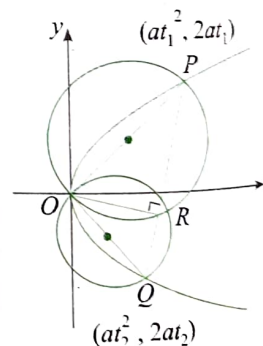
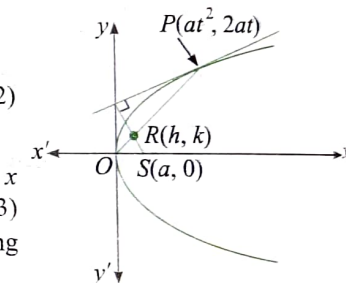
$$A_1 \equiv (0, at_1)$$

$$\text{and } B_1 \equiv (0, -at_1)$$

Area of trapezium

$$AA_1B_1B = \frac{1}{2}(AB + A_1B_1) \times OC$$

$$\therefore 24a^2 = \frac{1}{2}(4at_1 + 2at_1)(at_1^2)$$



or $t_1 = 0$ or $t_1 = 2$

$$\therefore A_1 \equiv (0, 2a)$$

If $\angle OSA_1 = \theta$, then

$$\tan \theta = \frac{2a}{a} = 2$$

$$\text{or } \theta = \tan^{-1}(2)$$

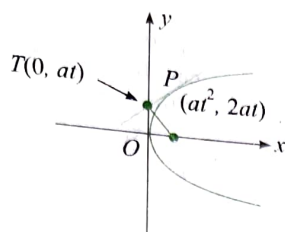
Thus, the required angle is $2 \tan^{-1} 2$.

53. (2) Let the middle point of P and T be (h, k) . Then,

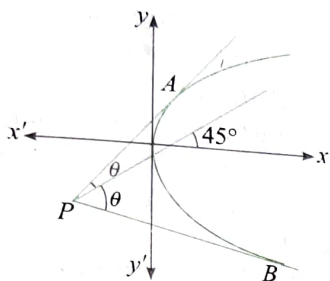
$$2h = at^2$$

$$\text{and } 2k = 3at$$

$$\therefore 2h = a \cdot \frac{4k^2}{9a^2}$$

The locus of (h, k) is $2y^2 = 9ax$.As $a = 2$, we get $y^2 = 9x$.

54. (4)



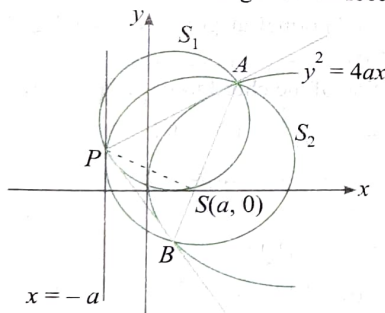
$$\text{Here, } \frac{1}{t_1} = \tan\left(\frac{\pi}{4} + \theta\right)$$

$$\text{and } \frac{1}{t_2} = \tan\left(\frac{\pi}{4} - \theta\right)$$

$$\text{So, } t_1 t_2 = 1$$

Therefore, the x-coordinate of P is $at_1 t_2$, i.e., a .

55. (3) Circle S_2 , taking focal chord AB as diameter will touch directrix at point P and circle S_1 , taking AP as diameter will pass through focus S (since AP subtends angle 90° at focus of parabola).



Hence, the common chord of the given circles is line AP (which is the intercept of tangent at point A between point A and directrix).

56. (3) Let the point of intersection be (α, β) .

Therefore, the chord of contact w.r.t. this point is

$$\beta y = 2x + 2\alpha$$

which is the same as $x + y = 2$. Therefore,

$$\alpha = \beta = -2$$

These values satisfy $y - x = 0$.

57. (4) The coordinates of the focus of the parabola $y^2 = 4ax$ are $(a, 0)$. The line $y - x - a = 0$ passes through this point. Therefore, it is a focal chord of the parabola. Hence, the tangent intersects at right angle.

58. (2) Clearly, P is the point of intersection of two perpendicular tangents to the parabola $y^2 = 8x$.

Hence, P must lie on the directrix $x + a = 0$ or $x + 2 = 0$.Therefore, $x = -2$.Hence, the point is $(-2, 0)$.

59. (4) Tangents $y = m_1 x + c$ and $y = m_2 x + c$ intersect at $(0, c)$ which lies on the directrix of the given parabola.

Hence, the tangents are perpendicular, for which $m_1 m_2 = -1$.

60. (1) $\frac{dy}{dx} = 2x - 5$

$$\therefore m_1 = \left(\frac{dy}{dx}\right)_{(2,0)} = 4 - 5 = -1$$

$$\text{and } m_2 = \left(\frac{dy}{dx}\right)_{(3,0)} = 6 - 5 = 1$$

$$\therefore m_1 m_2 = -1$$

Therefore, the angle between the tangents is $\pi/2$.

61. (2) Tangents at $A(t_1)$ and $A(t_2)$ are respectively,

$$t_1 y = x + at_1^2$$

$$t_2 y = x + at_2^2$$

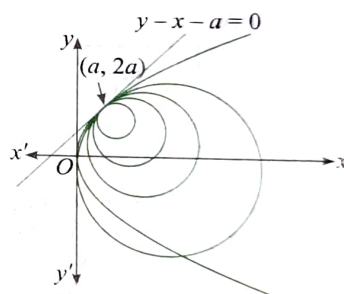
Tangents meet axis at $P_1(-at_1^2, 0)$ and $P_2(-at_2^2, 0)$.

$$\therefore SP_1 = a + at_1^2 \text{ and } SP_2 = a + at_2^2$$

Since tangents are perpendicular, $t_1 t_2 = -1$.

$$\begin{aligned} \therefore \frac{1}{SP_1} + \frac{1}{SP_2} &= \frac{1}{a + at_1^2} + \frac{1}{a + at_2^2} \\ &= \frac{1}{a + at_1^2} + \frac{1}{a + \frac{a}{t_1^2}} \\ &= \frac{1}{a + at_1^2} + \frac{t_1^2}{at_1^2 + a} \\ &= \frac{1}{a} \end{aligned}$$

62. (1)

The equation of tangent to the parabola at $(a, 2a)$ is

$$2ya = 2a(x + a)$$

$$\text{or } y - x - a = 0$$

The equation of the circle touching the parabola at $(a, 2a)$ is

$$(x - a)^2 + (y - 2a)^2 + \lambda(y - x - a) = 0$$

It passes through $(0, 0)$. Therefore,

$$a^2 + 4a^2 + \lambda(-a) = 0 \text{ or } \lambda = 5a$$

Thus, the required circle is $x^2 + y^2 - 7ax - ay = 0$.

Its radius is

$$\sqrt{\frac{49}{4}a^2 + \frac{a^2}{4}} = \frac{5}{\sqrt{2}}a$$

63. (3) Any point on the given parabola is $(t^2, 2t)$.

The equation of the tangent at $(1, 2)$ is $x - y + 1 = 0$.The image (h, k) of the point $(t^2, 2t)$ on $x - y + 1 = 0$ is given by

$$\frac{h - t^2}{1} = \frac{k - 2t}{-1} = \frac{2(t^2 - 2t + 1)}{1 + 1}$$

$$\therefore h = t^2 - t^2 + 2t - 1 = 2t - 1$$

$$\text{and } k = 2t + t^2 - 2t + 1 = t^2 + 1$$

Eliminating t from $h = 2t - 1$ and $k = t^2 + 1$, we get

$$(h + 1)^2 = 4(k - 1)$$

The required equation of reflection is $(x + 1)^2 = 4(y - 1)$.

64. (1) The area of triangle ABC is maximum if CD is maximum, because AB is fixed.

Therefore, the tangent drawn to the parabola at C should be parallel to AB .

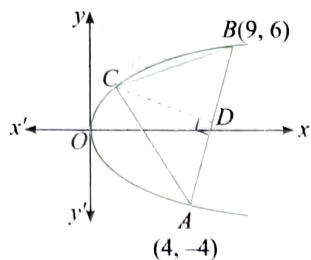
$$\text{Slope of } AB = \frac{6+4}{9-4} = 2$$

$$\text{For } y^2 = 4x,$$

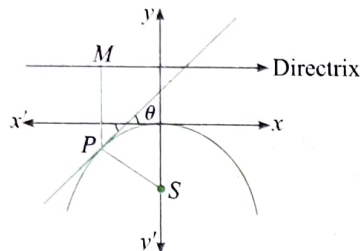
$$\frac{dy}{dx} = \frac{2}{y} = 2$$

$$\text{or } y = 1$$

$$\text{or } x = \frac{1}{4}$$



65. (2)

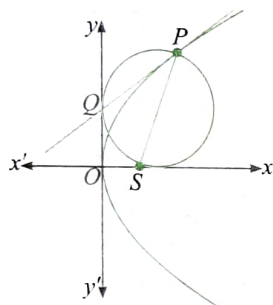


$$\text{Slope of line } = \lambda = \tan \theta$$

$$\therefore \tan(\angle MPS) = \tan 2\left(\frac{\pi}{2} - \theta\right) = \tan(\pi - 2\theta)$$

$$= -\tan 2\theta = \frac{2\lambda}{\lambda^2 - 1}$$

66. (2)



Tangent at P intersects the y -axis at $Q \equiv (0, at)$.

Also, circle with PS as diameter touches the y -axis at $(0, at)$. So, the y -axis is the tangent to the circumcircle of ΔPQS at Q .

67. (4) The equation of tangent at $P(t^2, 2t)$ is $ty = x + t^2$. It intersects the line $x = 0$ at $Q(0, t)$. Therefore,

$$\text{Area of } \Delta PQS = \frac{1}{2} \begin{vmatrix} 0 & t & 1 \\ 1 & 0 & 1 \\ t^2 & 2t & 1 \end{vmatrix}$$

$$= \frac{1}{2} (t + t^3)$$

$$\text{Now, } \frac{dA}{dt} = \frac{1}{2} (1 + 3t^2) > 0 \quad \forall t \in \mathbb{R}$$

Therefore, the area is maximum for $t = 2$. Hence,

$$\text{Maximum area} = \frac{1}{2} (2 + 8) = 5 \text{ sq. units}$$

68. (2) The parabolas $y = x^2 + 1$ and $x = y^2 + 1$ are symmetrical about $y = x$.

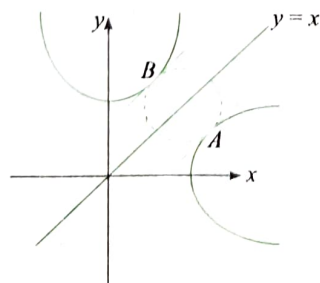
Therefore, the tangent at point A is parallel to $y = x$.

Therefore,

$$\frac{dy}{dx} = 2x \text{ or } 2x = 1$$

$$\text{or } x = \frac{1}{2} \text{ and } y = \frac{5}{4}$$

$$\therefore A \equiv \left(\frac{1}{2}, \frac{5}{4}\right)$$



$$\text{and } B \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$$

$$\text{Hence, Radius} = \frac{1}{2} \sqrt{\left(\frac{1}{2} - \frac{5}{4}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2}$$

$$= \frac{1}{2} \sqrt{\frac{9}{16} + \frac{9}{16}} = \frac{3}{8} \sqrt{2}$$

$$\therefore \text{Area} = \frac{9\pi}{32}$$

69. (3) Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2 = 4ax$. The tangents at P and Q intersect at $(at_1t_2, a(t_1 + t_2))$. Therefore,

$$x_1 = at_1t_2 \quad \text{and} \quad y_1 = a(t_1 + t_2)$$

$$\text{or } x_1 = -a \quad \text{and} \quad y_1 = a(t_1 + t_2)$$

$$[\because PQ \text{ is a focal chord, } t_1t_2 = -1]$$

The normals at P and Q intersect at

$$(2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2))$$

$$\therefore x_2 = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$

$$\text{and } y_2 = -at_1t_2(t_1 + t_2)$$

$$\text{or } x_2 = 2a + a(t_1^2 + t_2^2 - 1) = a + a(t_1^2 + t_2^2)$$

$$\text{and } y_2 = a(t_1 + t_2)$$

Clearly, $y_1 = y_2$.

70. (4) We have $a = 1$.

$$\text{Normal at } (m^2, -2m) \text{ is } y = mx - 2m - m^3.$$

Given that normal makes equal angle with the axes, its slope is $m = \pm 1$.

Therefore, point P is $(m^2, -2m) \equiv (1, \pm 2)$.

71. (3) $y = mx + c$ is a normal to $y^2 = 4ax$ if $c = -2am - am^3$, $y = -2x - \lambda$ (Given normal)

$$\therefore m = -2, a = -2$$

$$\text{or } -\lambda = -2am - am^3 = -2(-2)(-2) - (-2)(-2)^3 = -24$$

$$\text{or } \lambda = 24$$

72. (2) The slope of normal at points $P(t_1)$ and $Q(t_2)$ is $-t_1$ and $-t_2$, respectively.

The equation of the chord joining $P(t_1)$ and $Q(t_2)$ is

$$y - 2at_1 = \frac{2}{t_1 + t_2} (x - at_1^2)$$

$$\text{or } 2x - (t_1 + t_2)y + 2at_1t_2 = 0$$

$$\text{But } t_1t_2 = -1$$

Therefore, chord PQ is

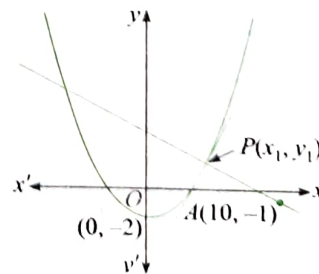
$$2x - (t_1 + t_2)y - 2a = 0$$

$$\text{or } (2x - 2a) - (t_1 + t_2)y = 0$$

which passes through the fixed point $(a, 0)$.

73. (4) $4y = x^2 - 8$

$$\text{or } 4 \frac{dy}{dx} = 2x$$



$$\therefore \text{Slope of normal} = -\frac{2}{x_1}$$

But

$$\text{Slope of normal} = \frac{y_1 + 1}{x_1 - 10}$$

$$\therefore \frac{y_1 + 1}{x_1 - 10} = -\frac{2}{x_1}$$

$$\text{or } x_1 y_1 + x_1 = -2x_1 + 20$$

$$\text{or } x_1 y_1 + 3x_1 = 20$$

Substituting

$$y_1 = \frac{x_1^2 - 8}{4}$$

(From the given equation)

$$\text{we get } x_1 \left(\frac{x_1^2 - 8}{4} + 3 \right) = 20$$

$$\text{or } x_1(x_1^2 + 4) = 80$$

$$\text{or } x_1^3 + 4x_1 - 80 = 0$$

which has one root, $x_1 = 4$.

Hence, $x_1 = 4$ and $y_1 = 2$.

Therefore, $P \equiv (4, 2)$.

Therefore, the equation of PA is

$$y + 1 = -\frac{1}{2}(x - 10)$$

$$\text{or } 2y + 2 = -x + 10$$

$$\text{or } x + 2y - 8 = 0$$

74. (2) The equations of tangent and normal at A are $yt = x + at^2$ and $y = -tx + 2at + at^3$, respectively.

Therefore, $B \equiv (-at^2, 0)$ and $D \equiv (2a + at^2, 0)$. If $ABCD$ is a rectangle, then the midpoints of BD and AC will be coincident. So,

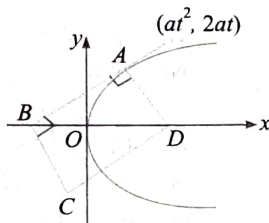
$$h + at^2 = 2a + at^2 - at^2,$$

$$k + 2at = 0$$

$$\text{or } h = 2a - at^2, t = -\frac{k}{2a}$$

Therefore, the locus is

$$x = 2a - \frac{y^2}{4a}$$

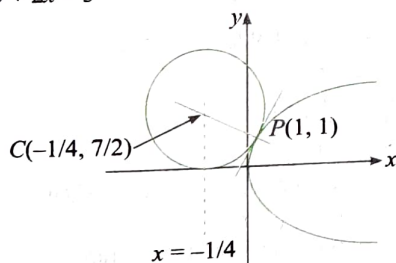


75. (3) The equation of normal at $P(1, 1)$ is

$$y - 1 = -2(x - 1)$$

$$\text{or } y + 2x = 3$$

(1)



The directrix of the parabola $y^2 = x$ is

$$x = -\frac{1}{4}$$

(2)

The center of the circle is the intersection of two normals to the circle, i.e., (1) and (2). Therefore, the center is $(-1/4, 7/2)$.

Hence, the radius of the circle is

$$\sqrt{\left(1 + \frac{1}{4}\right)^2 + \left(1 - \frac{7}{2}\right)^2} = \sqrt{\frac{25}{16} + \frac{25}{4}} = \frac{5\sqrt{5}}{4}$$

76. (2) Tangent to $y^2 = 4x$ in terms of m is

$$y = mx + \frac{1}{m}$$

(1)

Normal to $x^2 = 4by$ in terms of m is

$$y = mx + 2b + \frac{b}{m^2}$$

(2)

Equations (1) and (2) represent same line, then

$$\frac{1}{m} = 2b + \frac{b}{m^2}$$

$$\text{or } 2bm^2 - m + b = 0$$

For two different tangents

$$1 - 8b^2 > 0$$

$$\text{or } |b| < \frac{1}{\sqrt{8}}$$

77. (4) Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of m are

$$y = mx - 2am - am^3$$

$$\text{and } y = mx + 2b + \frac{b}{m^2}$$

For a common normal,

$$2b + \frac{b}{m^2} + 2am + am^3 = 0$$

$$\text{or } am^5 + 2am^3 + 2bm^2 + b = 0$$

This means there can be at most five common normals

78. (3) $4a = 2$ or $a = \frac{1}{2}$

Vertex $\equiv (-2, 3)$

So, the parabola is

$$(y - 3)^2 = 2(x + 2)$$

The equation of normal to the parabola is

$$(y - 3) = m(x + 2) - m - \frac{1}{2}m^3$$

$$\text{or } y = mx + m + 3 - \frac{1}{2}m^3$$

Also, given $y = 2x + k$. Here, $m = 2$.

$$k = m + 3 - \frac{1}{2}m^3$$

$$= 2 + 3 - \frac{1}{2}(2)^3 = 5 - 4 = 1$$

79. (2) Let $y_1 = 3 + \sqrt{1 - x_1^2}$ and $y_2 = \sqrt{4x_2}$

$$\text{or } x_1^2 + (y_1 - 3)^2 = 1 \text{ and } y_2^2 = 4x_2$$

Thus, (x_1, y_1) lies on the circle $x^2 + (y - 3)^2 = 1$.

Also, (x_2, y_2) lies on the upper half of the parabola $y^2 = 4x$.

Thus, the given expression is square of the shortest distance between the curves $x^2 + (y - 3)^2 = 1$ and $y^2 = 4x$.

Now, the shortest distance always occurs along the common normal to the curves and the normal to the circle passes through the center of the circle.

Normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$. It passes through $(0, 3)$.

Therefore, $m^3 + 2m + 3 = 0$, which has only one root, $m = -1$.

Hence, the corresponding point on the parabola is $(1, 2)$.

Thus, the required minimum distance is $\sqrt{1 + 1} - 1 = \sqrt{2} - 1$.

80. (4) Normals at points $(ap^2, 2ap)$, $(aq^2, 2aq)$, and $(ar^2, 2ar)$ are concurrent.

Hence, the points are co-normal points. Therefore,

$$p + q + r = 0$$

So, $px^2 + qx + r = 0$ has one root which is $x = 1$.

Therefore, the common root is 1, which also satisfies the second equation.

81. (3) Let $A_1 \equiv (2t_1^2, 4t_1)$ and $A_2 \equiv (2t_1^2, -4t_1)$.

Clearly, $\angle A_1OA = \pi/6$. Therefore,

$$\frac{2}{t_1} = \frac{1}{\sqrt{3}}$$

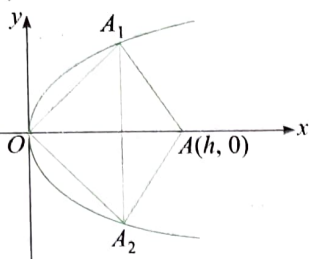
or $t_1 = 2\sqrt{3}$

The equation of normal at A_1 is

$$y = -t_1x + 4t_1 + 2t_1^3$$

or $h = 4 + 2t_1^2 = 4 + 2(12)$

$$= 28$$



82. (4) The ends of the latus rectum are $P(a, 2a)$ and $P'(a, -2a)$.

Point P has parameter $t_1 = 1$ and point P' has parameter $t_2 = -1$.

Normal at point P meets the curve again at point Q whose parameter

$$t_1' = -t_1 - \frac{2}{t_1} = -3$$

Normal at point P' meets the curve again at point Q' whose parameter

$$t_2' = -t_2 - \frac{2}{t_2} = 3$$

Hence, points Q and Q' have coordinates $(9a, -6a)$ and $(9a, 6a)$, respectively.

Hence, $QQ' = 12a$.

83. (4) Point $(\sin \theta, \cos \theta)$ lies on the circle $x^2 + y^2 = 1$ for $\forall \theta \in R$.

Now, three normals can be drawn to the parabola $y^2 = 4ax$ if $x = |2a|$ meets this circle.

Hence, we must have

$$\cos \theta > |2a|$$

or $0 < |2a| < 1$

or $0 < |a| < \frac{1}{2}$

or $a \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$

84. (4) Normal at point $P(t_1)$ meets the parabola again at point $R(t_3)$.

$$\text{Then } t_3 = -t_1 - \frac{2}{t_1}$$

Also, normal at point $Q(t_2)$ meets the parabola at the same point $R(t_3)$.

$$\text{Then } t_3 = -t_2 - \frac{2}{t_2}$$

Comparing these values of t_3 , we have

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$$\therefore t_1t_2 = 2$$

85. (2) $\tan \alpha = -t_1$ and $\tan \beta = -t_2$

Also, $t_2 = -t_1 - \frac{2}{t_1}$

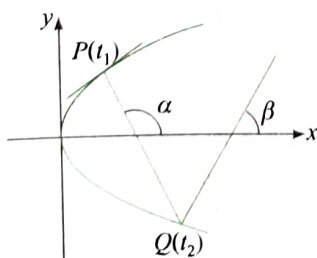
$$t_1t_2 + t_1^2 = -2$$

$$\tan \alpha \tan \beta + \tan^2 \alpha = -2$$

86. (3) $t_2 = -t_1 - \frac{2}{t_1}$ or $t_1t_2 = -t_1^2 - 2$

The equation of the line through P parallel to AQ is

$$y - 2at_1 = \frac{2}{t_2}(x - at_1^2)$$



Put $y = 0$. Then,

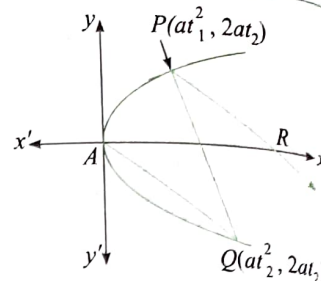
$$x = at_1^2 - at_1t_2$$

$$= at_1^2 - a(-2 - t_1^2)$$

$$= 2a + 2at_1^2$$

$$= 2(a + at_1^2)$$

= twice the focal distance of P



87. (1) A circle through three co-normal points of a parabola always passes through the vertex of the parabola. Hence, the circle through P, Q, R , and S out of which P, Q , and R are co-normals points will always pass through vertex $(2, 3)$ of the parabola.

88. (3) Normal at point $P(x_1, y_1) \equiv (at_1^2, 2at_1)$ meets the parabola at $P(at_1^2, 2at_1)$. So,

$$t = -t_1 - \frac{2}{t_1} \quad (1)$$

Normal at point $Q(x_2, y_2) \equiv (at_2^2, 2at_2)$ meets the parabola at $R(at_2^2, 2at_2)$. So,

$$t = -t_2 - \frac{2}{t_2} \quad (2)$$

From (1) and (2), we get

$$-t_1 - \frac{2}{t_1} = -t_2 - \frac{2}{t_2}$$

$$\therefore t_1t_2 = 2$$

Now given that $x_1 + x_2 = 4$. Therefore,

$$t_1^2 + t_2^2 = 4$$

or $(t_1 + t_2)^2 = 4 + 4 = 8$

or $|t_1 + t_2| = 2\sqrt{2}$

or $|y_1 + y_2| = 4\sqrt{2}$

89. (2) Let the concyclic points be t_1, t_2, t_3 , and t_4 . Then,

$$t_1 + t_2 + t_3 + t_4 = 0$$

Here, t_1 and t_3 are the feet of the normals. So,

$$t_2 = -t_1 - \frac{2}{t_1} \text{ and } t_4 = -t_3 - \frac{2}{t_3}$$

$$\therefore t_1 + t_2 = -\frac{2}{t_1} \text{ and } t_4 + t_3 = -\frac{2}{t_3}$$

Adding, we get

$$-2\left(\frac{1}{t_1} + \frac{1}{t_3}\right) = 0$$

or $t_1 + t_3 = 0$

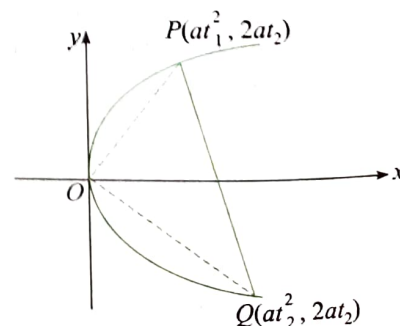
Therefore, the point of intersection of tangents at t_1 and t_3 is

$$(at_1t_3, a(t_1 + t_3)) \equiv (at_1t_3, 0).$$

This point lies on the axis of the parabola.

90. (1) Normal at $P(t_1)$ meets at $Q(t_2)$. So,

$$t_2 = -\frac{2}{t_1} - t_1$$



$$|t_2| \geq 2\sqrt{2}$$

For minimum length of OQ , $|t_2|$ should be minimum. Therefore,

$$|t_2| = 2\sqrt{2} \text{ or } t_2 = \pm 2\sqrt{2}$$

If $t_2 = -2\sqrt{2}$, then $t_1 = \sqrt{2}$.

$$\text{Slope of } OQ = \frac{2}{t_2} = m_1$$

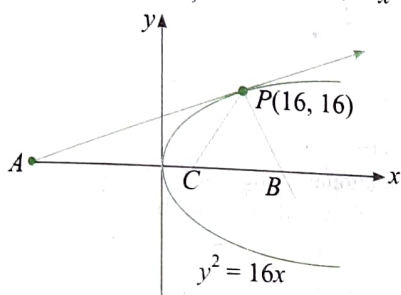
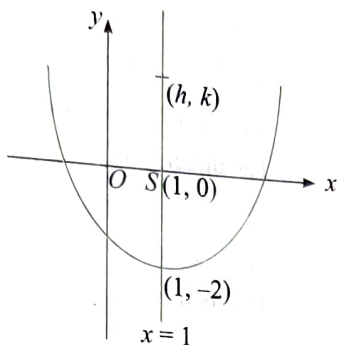
$$\text{Slope of } OP = \frac{2}{t_1} = m_2$$

$$\therefore m_1 m_2 = -1$$

So, $\triangle OPQ$ is a right-angled triangle.

91. (3) The axis of the parabola is $x = 1$. Any point on it is $(1, k)$. Now, the distance of $(1, k)$ from $(1, -2)$ should be more than the semi-latus rectum, and $(1, k)$ should be inside the parabola. Hence, $k > 2$.

92. (4) By property, the center of the circle coincides with the focus of the parabola. So,



$$C \equiv (4, 0)$$

$$\tan \alpha = \text{Slope of } PC = \frac{16}{12}$$

$$\text{or } \alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

93. (3) Let the equation of any normal be $y = -tx + 2t + t^3$.

Since it passes through the point $(15, 12)$, we have

$$12 = -15t + 2t + t^3$$

$$\text{or } t^3 - 13t - 12 = 0$$

One root is -1 . Then,

$$(t+1)(t^2 - t - 12) = 0$$

$$\text{or } t = 1, -3, 4$$

Therefore, the co-normal points are $(1, -2)$, $(9, -6)$, and $(16, 8)$.

Therefore, the centroid is $(26/3, 0)$.

94. (2) Solving the line $y = x - 1$ and the parabola $y^2 = 4x$, we have

$$(x-1)^2 = x$$

$$\text{or } x^2 - 6x + 1 = 0$$

$$\text{or } x = 3 \pm \sqrt{8}$$

$$\therefore y = 2 \pm \sqrt{8}$$

Suppose point D is (x_3, y_3) . Then,

$$y_1 + y_2 + y_3 = 0$$

$$\text{or } 2 + \sqrt{8} + 2 - \sqrt{8} + y_3 = 0$$

$$\text{or } y_3 = -4$$

Then $x_3 = 4$.

Therefore, the point is $(4, 4)$.

95. (3) The equation of normal is $y = mx - 2am - am^3$.

Putting $y = 0$, we get

$$x_1 = 2a + am_1^2$$

$$x_2 = 2a + am_2^2$$

$$\text{and } x_3 = 2a + am_3^2$$

where x_1, x_2 , and x_3 are the intercepts on the axis of the parabola.

The normal passes through (h, k) . Therefore,

$$am^3 + (2a - h)m + k = 0$$

It has roots m_1, m_2 , and m_3 which are the slopes of the normals.

Therefore,

$$m_1 + m_2 + m_3 = 0$$

$$\text{and } m_1 m_2 + m_2 m_3 + m_3 m_1 = \frac{2a - h}{a}$$

$$\begin{aligned} \therefore m_1^2 + m_2^2 + m_3^2 &= (m_1 + m_2 + m_3)^2 - 2(m_1 m_2 + m_2 m_3 + m_3 m_1) \\ &= -\frac{2(2a - h)}{a} \end{aligned}$$

$$\text{or } x_1 + x_2 + x_3 = 6a - 2(2a - h) = 2(h + a)$$

Multiple Correct Answers Type

1. (2), (4)

The given parabola is

$$x^2 - ky + 3 = 0$$

$$\text{or } x^2 = k\left(y - \frac{3}{k}\right)$$

$$\text{Let } x = Y, y - \frac{3}{k} = X$$

Then the parabola is

$$Y^2 = kX$$

whose focus is $(0, k/4)$.

Therefore, the focus of

$$x^2 = k\left(y - \frac{3}{k}\right)$$

is

$$\left(0, \frac{3}{k} + \frac{k}{4}\right) \equiv (0, 2)$$

$$\therefore \frac{3}{k} + \frac{k}{4} = 2$$

$$\text{or } 12 + k^2 = 8k$$

$$\text{or } k^2 - 8k + 12 = 0$$

$$\text{or } (k-6)(k-2) = 0$$

$$\text{or } k = 2, 6$$

2. (1), (4)

We have equation of parabola,

$$y^2 = kx - 8 \text{ or } y^2 = k\left(x - \frac{8}{k}\right)$$

Equation of directrix is

$$x - \frac{8}{k} = -\frac{k}{4} \text{ or } x = -\frac{k}{4} + \frac{8}{k}$$

Given equation of directrix is $x = 1$.

$$\therefore \frac{8}{k} - \frac{k}{4} = 1$$

$$\Rightarrow k^2 + 4k - 32 = 0$$

$$\Rightarrow (k-4)(k+8) = 0$$

$$\Rightarrow k = -8, 4$$

3. (1), (3)

Given that the extremities of the latus rectum are $(1, 1)$ and $(1, -1)$. Then,

$$4a = 2 \text{ or } a = \frac{1}{2}$$

Also, the focus of the parabola is $(1, 0)$.

Hence, the vertex can be $(1/2, 0)$ or $(3/2, 0)$.

Therefore, the equations of the parabola can be

$$y^2 = 2\left(x - \frac{1}{2}\right)$$

or $y^2 = -2\left(x - \frac{3}{2}\right)$

i.e., $y^2 = 2x - 1$

or $y^2 = 3 - 2x$

4. (1), (3)

The two curves are symmetric about the line $y = x$.

Hence, they touch each other on $y = x$.

So, the point of contact is (α, α) .

From any of the two equations, we get

$$\alpha = a\alpha^2 + a\alpha + \frac{1}{24}$$

or $24a\alpha^2 + 24\alpha(a - 1) + 1 = 0$

This equation should have identical roots.

$$\Rightarrow D = 0$$

$$\Rightarrow (24)^2(a - 1)^2 - 4(24a) = 0$$

$$\Rightarrow 6a^2 - 13a + 6 = 0$$

$$\Rightarrow (2a - 3)(3a - 2) = 0$$

$$\Rightarrow a = 3/2, 2/3$$

5. (1), (2), (3)

- (1) Focus and equation of tangent at vertex are given.

Directrix is parallel to tangent at vertex.

Also, distance between focus and directrix is same as distance between directrix and tangent at vertex.

So, equation of directrix can be determined.

Therefore, parabola is unique.

- (2) Focus and vertex are given.

Distance between focus and vertex is ' a '.

Also, distance between vertex and directrix is ' a '.

Directrix is perpendicular to the axis which passes through focus and vertex.

So, equation of directrix can be determined.

Therefore, parabola is unique.

- (3) Equation of directrix and vertex are given.

So, focus can be found.

Therefore, parabola is unique.

- (4) Equation of directrix and equation of tangent at vertex are given.

Here, only distance between these two lines can be found, four times of which is latus rectum.

But focus cannot be located.

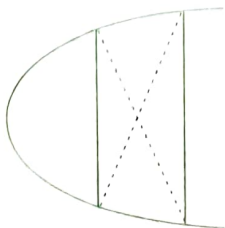
Therefore, parabola is not unique.

6. (1), (2)

As a circle can intersect a parabola at four points, the quadrilateral may be cyclic.

The diagonals of the quadrilateral may be equal as the quadrilateral may be an isosceles trapezium.

A rectangle cannot be inscribed in a parabola. So, option (3) is wrong.



7. (1), (2), (3), (4)

Any point on the parabola is $P(at^2, 2at)$.

Therefore, the midpoint of $S(a, 0)$ and $P(at^2, 2at)$ is

$$R\left(\frac{a + at^2}{2}, at\right) \equiv (h, k)$$

$$\therefore h = \frac{a + at^2}{2}, k = at$$

Eliminate t , i.e.,

$$2x = a\left(1 + \frac{y^2}{a^2}\right) = a + \frac{y^2}{a}$$

i.e., $2ax = a^2 + y^2$

i.e., $y^2 = 2a\left(x - \frac{a}{2}\right)$

It is a parabola with vertex at $(a/2, 0)$ and latus rectum $2a$.

The directrix is

$$x - \frac{a}{2} = -\frac{a}{2}$$

i.e., $x = 0$

The focus is

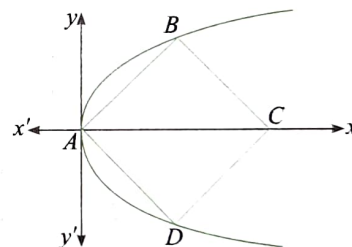
$$x - \frac{a}{2} = \frac{a}{2}$$

i.e., $x = a$

i.e., $(a, 0)$

8. (1), (2), (3), (4)

AC is one diagonal along the x -axis. Then the other diagonal is BD , where both B and D lie on the parabola. Also,



$$\text{Slope of } AB = \tan \frac{\pi}{4} = 1$$

If B is $(at^2, 2at)$, then

$$\text{Slope of } AB = \frac{2at}{at^2} = \frac{2}{t} = 1$$

$$\therefore t = 2$$

Therefore, B is $(4a, 4a)$ and hence, D is $(4a, -4a)$.

Clearly, C is $(8a, 0)$.

9. (1), (2), (4)

Any point on $x + y = 1$ can be taken as $(t, 1 - t)$.

The equation of chord with this as midpoint is

$$y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$$

It passes through $(a, 2a)$. So,

$$t^2 - 2t + 2a^2 - 2a + 1 = 0$$

This should have two distinct real roots. So,

$$\text{Discriminant} > 0 \text{ i.e., } a^2 - a < 0$$

$$0 < a < 1 \text{ or } 0 < 4a < 4$$

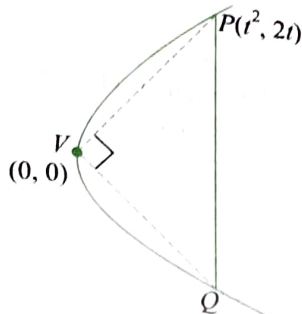
So, length of latus rectum lies in $(0, 4)$.

10. (1), (2)

$$\text{Slope of } PV = \frac{2t - 0}{t^2 - 0} = \frac{2}{t}$$

Therefore, the equation of QV is

$$y = -\frac{t}{2}x$$



Solving it with $y^2 = 4x$, we get $Q \equiv (16/t^2, -8/t)$. Now,

$$\ar(\Delta PVQ) = \frac{1}{2} PV \cdot VQ = 20 \text{ (Given)}$$

$$\therefore PV^2 \cdot VQ^2 = 40^2$$

$$\text{or } \{(t^2)^2 + (2t)^2\} \left\{ \left(\frac{16}{t^2} \right)^2 + \left(\frac{-8}{t} \right)^2 \right\} = 40^2$$

$$\text{or } t^2(4 + t^2) \frac{1}{t^2} \left(\frac{256}{t^2} + 64 \right) = 40^2$$

$$\text{or } \frac{256 \times 4}{t^2} + 256 + 256 + 64t^2 = 40^2$$

$$\text{or } (t^2 - 16)(t^2 - 1) = 0$$

$$\therefore t = \pm 4, \pm 1$$

11. (2), (3)

Let line intersects parabola at $A(x_1, y_1)$ and $B(x_2, y_2)$.

Solving given parabola and line, we have

$$x^2 = a(1 + 2x)$$

$$\therefore x^2 - 2ax - a = 0$$

$$\text{So, } x_1 + x_2 = 2a \text{ and } x_1 x_2 = -a$$

$$\text{Now, } AB^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= (x_1 - x_2)^2 + ((2x_1 + 1) - (2x_2 + 1))^2$$

$$= 5(x_1 - x_2)^2$$

$$= 5[(x_1 + x_2)^2 - 4x_1 x_2] = 5(4a^2 + 4a)$$

$$\text{So, } 5(4a^2 + 4a) = 40$$

$$\therefore a^2 + a - 2 = 0$$

$$\therefore (a - 1)(a + 2) = 0$$

$$\therefore a = -2, 1$$

12. (1), (3)

The line $y = 2x + c$ is a tangent to $x^2 + y^2 = 5$.

If $c^2 = 25$, then $c = \pm 5$.

Let the equation of the parabola be $y^2 = 4ax$. Then

$$\frac{a}{2} = \pm 5$$

$$\text{or } a = \pm 10$$

So, the equation of the parabola is $y^2 = \pm 40x$.

Also, the equations of the directrices are $x = \pm 10$.

13. (2), (3), (4)

Let $(x_1, y_1) \equiv (at^2, 2at)$.

Tangent at this point is $ty = x + at^2$.

Any point on this tangent is $(h, (h + at^2)/t)$.

The chord of contact of this point with respect to the circle

$x^2 + y^2 = a^2$ is

$$hx + \left(\frac{h + at^2}{t} \right) y = a^2$$

$$\text{or } (aty - a^2) + h\left(x + \frac{y}{t}\right) = 0$$

which is a family of straight lines passing through the point of intersection of

$$ty - a = 0 \text{ and } x + \frac{y}{t} = 0$$

So, the fixed point is $(-a/t^2, a/t)$. Therefore,

$$x_2 = -\frac{a}{t^2}, y_2 = \frac{a}{t}$$

Clearly, $x_1 x_2 = -a^2, y_1 y_2 = 2a^2$

$$\text{Also, } \frac{x_1}{x_2} = -t^4$$

$$\text{and } \frac{y_1}{y_2} = 2t^2$$

$$\text{or } 4 \frac{x_1}{x_2} + \left(\frac{y_1}{y_2} \right)^2 = 0$$

14. (1), (3)

Let the possible point be $(t^2, 2t)$. The equation of tangent at this point is $yt = x + t^2$.

It must pass through $(6, 5)$, since the normal to the circle always passes through its center. Therefore,

$$t^2 - 5t + 6 = 0$$

$$\text{or } t = 2, 3$$

So, the possible points are $(4, 4)$ and $(9, 6)$.

15. (1), (2), (3)

The equation of tangent to the parabola $y^2 = 8x$ having slope m is

$$y = mx + \frac{2}{m}$$

Options (1), (2), and (3) are tangents for $m = 1, 3$, and $-1/2$, respectively.

16. (1), (2), (3), (4)

Given parabola is

$$(y - 2)^2 = 4(x - 1)$$

Equation of tangent having slope m is

$$(y - 2) = m(x - 1) + \frac{1}{m}$$

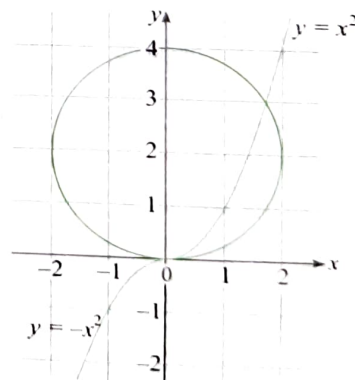
$$\text{or } m^2(x - 1) - m(y - 2) + 1 = 0$$

$$\text{or } (-m)^2(x - 1) + (-m)(y - 2) + 1 = 0$$

Comparing with given equation of tangent,

$$k \in \mathbb{R} - \{0\}$$

17. (1), (2), (3)



$$y = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$

Clearly, $y = 0$ is one of the common tangents.

Let the line be

$$y = mx + c$$

If it touches $y = x^2$, $x \geq 0$, then $mx + c = x^2$ has equal roots, i.e.,

$$m^2 + 4c = 0 \text{ or } c = \frac{-m^2}{4} \quad (2)$$

Also, the line touches the circle. Therefore,

$$\frac{|0 - 2 + c|}{\sqrt{m^2 + 1}} = 2 \quad (3)$$

$$\text{or } 4 - 4c + c^2 = 4(1 - 4c)$$

$$c^2 + 12c = 0$$

$$\text{i.e., } c = 0 \text{ or } -12$$

$$\text{For } c = 0, m = 0$$

$$\text{and for } c = -12, m = \sqrt{48}$$

Therefore, the equation of tangent is $y = \sqrt{48}x - 12$.

If line (1) touches $y = -x^2$, $x < 0$, then the equation $mx + c = -x^2$ has equal roots. Therefore,

$$D = 0$$

$$\text{or } m^2 - 4c = 0 \quad (4)$$

From (3) and (4), we get

$$c^2 - 20c = 0$$

$$\therefore c = 0 \text{ or } c = 20$$

$$\text{For } c = 20, m = \sqrt{80}$$

Therefore, the equation of tangent is $y = \sqrt{80}x + 20$.

18. (1), (2)

If $y = mx + c$ is tangent to $y = x^2$, then $x^2 - mx - c = 0$ has equal roots. So,

$$m^2 + 4c = 0$$

$$\text{or } c = -\frac{m^2}{4}$$

So, the tangent to $y = x^2$ is

$$y = mx - \frac{m^2}{4}$$

Since this is also tangent to $y = -(x - 2)^2$,

$$mx - \frac{m^2}{4} = -x^2 + 4x - 4$$

has equal roots. So,

$$x^2 + (m - 4)x + \left(4 - \frac{m^2}{4}\right) = 0$$

has equal roots. So,

$$(m - 4)^2 - 4\left(4 - \frac{m^2}{4}\right) = 0$$

$$\text{or } m^2 - 8m + 16 + m^2 - 16 = 0$$

$$\text{or } m = 0, 4$$

So, $y = 0$ and $y = 4x - 4$ is the tangent.

19. (1), (2), (3)

For parabola $y^2 = 4ax$, equation of tangent at $A(at^2, 2at)$ is

$$ty = x + at^2$$

$$\therefore C \equiv (0, at), B \equiv \left(-a, at - \frac{a}{t}\right)$$

$$AC = at\sqrt{1+t^2}; BC = \frac{a}{t}\sqrt{1+t^2}; (CS)^2 = a^2(1+t^2)$$

$$\Rightarrow AC \cdot BC = (CS)^2$$

$$\text{Slope of } CS \times \text{Slope of } AB = \frac{0-at}{a-0} \times \frac{1}{t} = -1$$

Hence, CS is perpendicular to AB

$$CS = \text{Distance of } S(2, 0) \text{ and } x + y + 2 = 0$$

$$= \frac{|2 + 2|}{\sqrt{2}} = 2\sqrt{2}$$

$$\therefore AC \cdot BC = (CS)^2 = 8$$

20. (1), (3), (4)

The equation of normal to the parabola $y^2 = 12x$ having slope m is $y = mx - 6m - 3m^3$. Options (1), (3), and (4) are normal for $m = -1, -2$, and 3 , respectively.

21. (3), (4)

$$t_2 = -t_1 - \frac{2}{t_1}$$

$$\text{Also, } \frac{2at_1}{at_1^2} \times \frac{2at_2}{at_2^2} = -1$$

$$\text{or } t_1 t_2 = -4$$

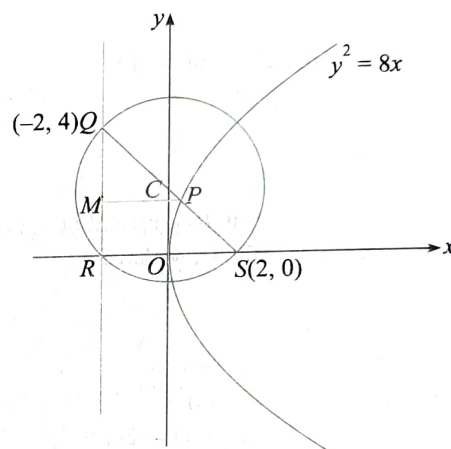
$$\therefore \frac{-4}{t_1} = -t_1 - \frac{2}{t_1}$$

$$\text{or } t_1^2 + 2 = 4 \text{ and } t_1 = \pm\sqrt{2}$$

So, the point can be $(2a, \pm 2\sqrt{2}a)$.

22. (1), (2), (3)

Image of S w.r.t. point $C(0, 2)$ is $Q(-2, 4)$. Thus, SQ is diameter. So circle must pass through R (foot of directrix on axis of parabola)



Equation of CS is $x + y = 2$ and parabola is $y^2 = 8x$.

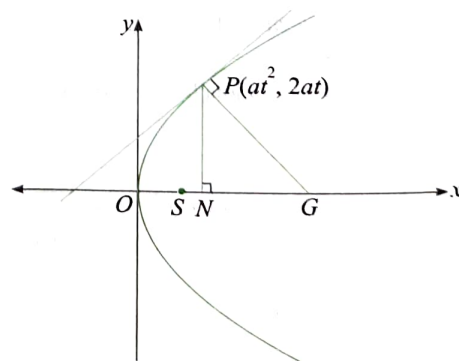
Solving, we get x -coordinate of P as $6 - 4\sqrt{2}$.

$$\therefore PM = 8 - 4\sqrt{2}$$

$$CP = CS - SP = 2\sqrt{2} - (8 - 4\sqrt{2}) = 6\sqrt{2} - 8$$

Slope of $CQ \times$ Slope of $CR = -1$

23. (2), (4)



$y^2 = 4ax$ is the given parabola and N divides SG internally in the ratio $1 : 3$.

$$S \equiv (a, 0) \text{ and } N \equiv (at^2, 0)$$

Equation of the normal PG is

$$y = -tx + 2at + at^3$$

It meets x -axis at $y = 0$.

$$\therefore tx = 2at + at^3$$

$$\therefore x = 2a + at^2$$

$$\therefore G \equiv (2a + at^2, 0)$$

N divides SG internally in the ratio $1 : 3$.

$$\therefore at^2 = \frac{2a + at^2 + 3a}{4}$$

$$\Rightarrow 3at^2 = 5a \Rightarrow t = \pm \sqrt{\frac{5}{3}}$$

24. (1), (2)

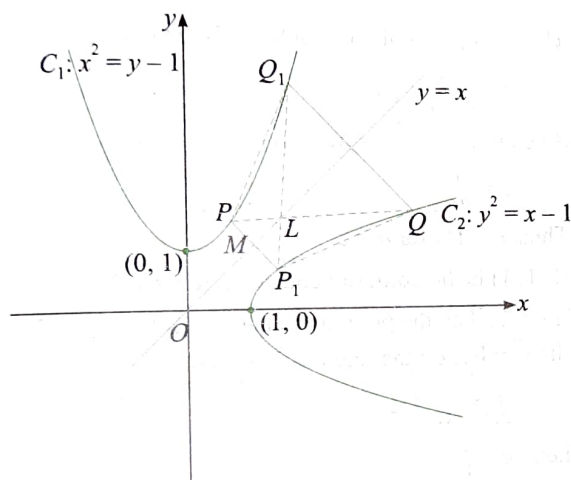
$$\text{Given that } C_1 : x^2 = y - 1$$

$$C_2 : y^2 = x - 1$$

Here, C_1 and C_2 are symmetrical about the line $y = x$.

Let us take points $P(x_1, x_1^2 + 1)$ on C_1 and $Q(y_2^2 + 1, y_2)$ on C_2 .

Then image of P in $y = x$ is $P_1(x_1^2 + 1, x_1)$ on C_2 and image of Q in $y = x$ is $Q_1(y_2^2 + 1, y_2)$ on C_1 .



Now, PP_1 and QQ_1 both are perpendicular to mirror line $y = x$.

Also, M is midpoint of PP_1 . ($\because P_1$ is mirror of P in $y = x$)

$$\therefore PM = \frac{1}{2} PP_1$$

In $\triangle PML$, $PL > PM$

$$\Rightarrow PL > \frac{1}{2} PP_1 \quad (1)$$

$$\text{Similarly, } LQ > \frac{1}{2} QQ_1 \quad (2)$$

Adding (1) and (2), we get

$$PL + LQ > \frac{1}{2} (PP_1 + QQ_1)$$

$$\Rightarrow PQ > \frac{1}{2} (PP_1 + QQ_1)$$

So, PQ is more than mean of PP_1 and QQ_1 .

So, $PQ \geq \min(PP_1, QQ_1)$

Let $\min(PP_1, QQ_1) = PP_1$.

$$\begin{aligned} \text{Then } PQ^2 &\geq PP_1^2 \\ &= (x_1^2 + 1 - x_1)^2 + (x_1^2 + 1 - x_1)^2 \\ &= 2(x_1^2 + 1 - x_1)^2 = f(x_1) \end{aligned}$$

$$\begin{aligned} \Rightarrow f'(x_1) &= 4(x_1^2 + 1 - x_1)(2x_1 - 1) \\ &= 4 \left(\left(x_1 - \frac{1}{2} \right)^2 + \frac{3}{4} \right) (2x_1 - 1) \end{aligned}$$

Therefore, $f'(x_1) = 0$ when $x_1 = \frac{1}{2}$.

Also, $f'(x_1) < 0$ if $x_1 < \frac{1}{2}$

and $f'(x_1) > 0$ if $x_1 > \frac{1}{2}$

Thus, $f(x_1)$ is minimum when $x_1 = \frac{1}{2}$.

Thus, at $x_1 = \frac{1}{2}$ point P is P_0 on C_1 .

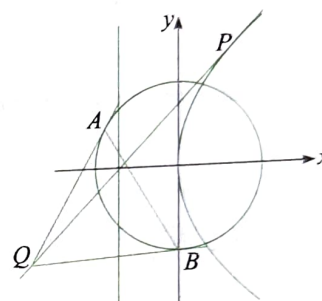
$$P_0 \equiv \left(\frac{1}{2}, \left(\frac{1}{2} \right)^2 + 1 \right) \equiv \left(\frac{1}{2}, \frac{5}{4} \right)$$

Similarly, Q_0 on C_2 will be image of P_0 with respect to $y = x$.

$$\therefore Q_0 \equiv \left(\frac{5}{4}, \frac{1}{2} \right)$$

Linked Comprehension Type

For Problems 1–3



1. (3) The equation of the tangent at point P of the parabola $y^2 = 8x$ is

$$yt = x + 2t^2 \quad (1)$$

The equation of the chord of contact of the circle $x^2 + y^2 = 8$ w.r.t. $Q(\alpha, \beta)$ is

$$x\alpha + y\beta = 8 \quad (2)$$

$Q(\alpha, \beta)$ lies on (1). Hence,

$$\beta t = \alpha + 2t^2 \quad (3)$$

$$\therefore x\alpha + y \left(\frac{\alpha}{t} + 2t \right) - 8 = 0 \quad [\text{From (2) and (3)}]$$

$$\text{or } 2(ty - 4) + \alpha \left(x + \frac{y}{t} \right) = 0$$

For point of concurrency,

$$x = -\frac{y}{t} \text{ and } y = \frac{4}{t}$$

Therefore, the locus is $y^2 + 4x = 0$.

2. (4) The required point (x_1, y_1) will lie on the director circle of the given circle as well as on the directrix of parabola. Therefore,

$$x_1^2 + y_1^2 = 16 \text{ and } x_1 + 2 = 0$$

$$\text{or } 4 + y_1^2 = 16$$

$$\text{or } y = \pm 2\sqrt{3}$$

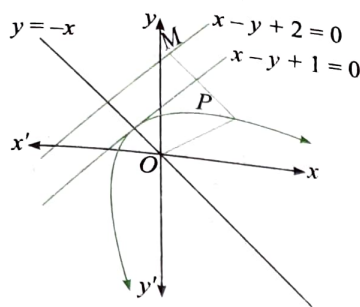
Therefore, the points are $(-2, \pm 2\sqrt{3})$.

3. (1) The equation of the circumcenter of $\triangle AQB$ is

$$x^2 + y^2 - 4 + \lambda(x\alpha + y\beta - 8) = 0$$

Because it passes through $(0, 0)$, i.e., the center of the circle,

$$\lambda = -\frac{1}{2}$$



The distance between the focus and the tangent at the vertex is

$$\frac{|0 - 0 + 1|}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$$

The directrix is the line parallel to the tangent at vertex and at a distance $2 \times 1/\sqrt{2}$ from the focus.

Let the equation of the directrix be

$$x - y + \lambda = 0$$

where $\frac{\lambda}{\sqrt{1^2 + 1^2}} = \frac{2}{\sqrt{2}}$

$$\therefore \lambda = 2$$

Let $P(x, y)$ be any moving point on the parabola. Then, $OP = PM$

$$\text{or } x^2 + y^2 = \left(\frac{x - y + 2}{\sqrt{1^2 + 1^2}} \right)^2$$

$$\text{or } 2x^2 + 2y^2 = (x - y + 2)^2$$

$$\text{or } x^2 + y^2 + 2xy - 4x + 4y - 4 = 0$$

Latus rectum length = $2 \times$ (Distance of focus from directrix)

$$= 2 \left| \frac{0 - 0 + 2}{\sqrt{1^2 + 1^2}} \right| = 2\sqrt{2}$$

Solving the parabola with the x -axis, we get

$$x^2 - 4x - 4 = 0$$

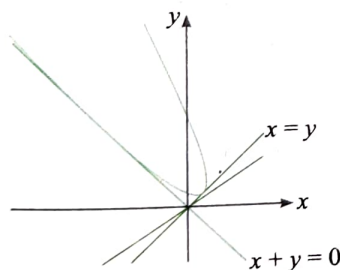
$$\text{or } x = \frac{4 \pm \sqrt{32}}{2} = 2 \pm 2\sqrt{2}$$

Therefore, the length of chord on the x -axis is $4\sqrt{2}$.

Since the chord $3x + 2y = 0$ passes through the focus, it is focal chord.

Hence, tangents at the extremities of chord are perpendicular.

19. (1), 20. (2), 21. (3)



We know that the foot of perpendicular from the focus upon a tangent lies on the tangent at the vertex of the parabola.

Now, if the foot of perpendicular of $(2, 3)$ on the line $x - y = 0$ is (x_1, y_1) , then

$$\frac{x_1 - 2}{1} = \frac{y_1 - 3}{-1} = \frac{-(2 - 3)}{2}$$

$$\text{or } x_1 = \frac{5}{2} \text{ and } y_1 = \frac{5}{2}$$

If the foot of perpendicular of $(2, 3)$ on the line $x + y = 0$ is (x_2, y_2) , then

$$\frac{x_2 - 2}{1} = \frac{y_2 - 3}{1} = \frac{2 + 3}{-2}$$

$$\text{or } x_2 = -\frac{1}{2} \text{ and } y_2 = \frac{1}{2}$$

Now, the tangent at the vertex passes through the points $(5/2, 5/2)$ and $(-1/2, 1/2)$. Then, its equation is

$$y - \frac{1}{2} = \frac{2}{3} \left(x + \frac{1}{2} \right)$$

$$\text{or } 4x - 6y + 5 = 0$$

The length of latus rectum of the parabola is

$$4 \times (\text{Distance of focus from tangent at vertex})$$

$$= 4 \times \left| \frac{8 - 18 + 5}{\sqrt{5^2 + 12^2}} \right| = \frac{10}{\sqrt{13}}$$

Also, the distance between the focus and the tangent at vertex is $5/\sqrt{13}$.

We know that

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$$

where a is $1/4$ th of the length of latus rectum. Therefore,

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2\sqrt{13}}{5}$$

22. (4), 23. (2), 24. (1)

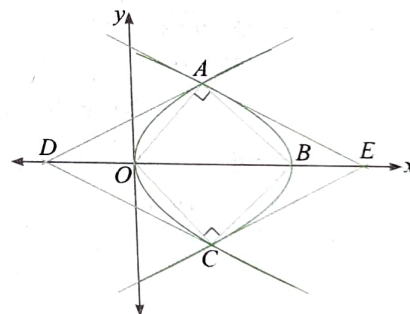
Solving the given parabolas, we have

$$-8(x - a) = 4x$$

$$\text{or } x = \frac{2a}{3}$$

Therefore, the points of intersection are $(2a/3, \pm\sqrt{8a/3})$.

Now, $OABC$ is cyclic quadrilateral.



Hence, $\angle OAB$ must be a right angle. So,

$$\text{Slope of } OA \times \text{Slope of } AB = -1$$

$$\text{or } \frac{\sqrt{8a/3}}{2a/3} \times \frac{\sqrt{8a/3}}{a - (2a/3)} = -1$$

$$\text{or } a = 12$$

Therefore, the coordinates of A and B are $(8, 4\sqrt{2})$ and $(8, -4\sqrt{2})$, respectively. So,

$$\text{Length of common chord} = 8\sqrt{2}$$

$$\text{Area of quadrilateral} = \frac{1}{2} OB \times AC$$

$$= \frac{1}{2} \times 12 \times 8\sqrt{2}$$

$$= 48\sqrt{2}$$

Tangent to the parabola $y^2 = 4x$ at $(8, 4\sqrt{2})$ is $4\sqrt{2}y = 2(x + 8)$ or $x - 2\sqrt{2}y + 8 = 0$, which meets the x -axis at $D(-8, 0)$.

Tangent to the parabola $y^2 = -8(x - 12)$ at $(8, 4\sqrt{2})$ is $4\sqrt{2}y = -4(x + 8) + 96$ or $x + \sqrt{2}y - 16 = 0$, which meets the x -axis at $E(16, 0)$. Hence,

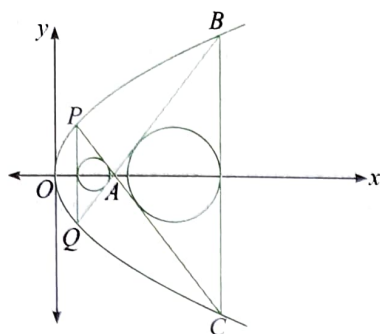
$$\text{Area of quadrilateral } DAEC = \frac{1}{2} DE \times AC$$

$$= \frac{1}{2} \times 24 \times 8\sqrt{2}$$

$$= 96\sqrt{2}$$

25. (2), 26. (3), 27. (4)

For $y^2 = 4x$, the coordinates of the ends of latus rectum are $P(1, 2)$ and $Q(1, -2)$.



ΔPAQ is isosceles right-angled. Therefore, the slope of PA is -1 and its equation is $y - 2 = -(x - 1)$, i.e., $x + y - 3 = 0$.

Similarly, the equation of line QB is $x - y - 3 = 0$.

Solving $x + y - 3 = 0$ with the parabola $y^2 = 4x$, we have

$$(3 - x)^2 = 4x, \text{ i.e., } x^2 - 10x + 9 = 0$$

$$\therefore x = 1, 9$$

Therefore, the coordinates of B and C are $(9, -6)$ and $(9, 6)$, respectively.

$$\begin{aligned} \text{Area of trapezium } PBCQ &= \frac{1}{2} \times (12 + 4) \times 8 \\ &= 64 \text{ sq. units} \end{aligned}$$

Let the circumcenter of trapezium $PBCQ$ be $T(h, 0)$. Then,
 $PT = BT$

$$\text{or } \sqrt{(h-1)^2 + 4} = \sqrt{(h-9)^2 + 36}$$

$$\text{or } -2h + 5 = -18h + 81 + 36$$

$$\text{or } 16h = 112$$

$$\text{or } h = 7$$

Hence, radius is $\sqrt{40} = 2\sqrt{10}$.

Let the inradius of ΔAPQ be r_1 . Then,

$$\begin{aligned} r_1 &= \frac{\Delta_1}{s_1} \\ &= \frac{\frac{1}{2} \times 4 \times 2}{\frac{1}{2}(4 + 2\sqrt{4+4})} \\ &= \frac{2}{1 + \sqrt{2}} = 2(\sqrt{2} - 1) \end{aligned}$$

Let the inradius of ΔABC be r_2 . Then,

$$\begin{aligned} r_2 &= \frac{\Delta_2}{s_2} \\ &= \frac{\frac{1}{2} \times 12 \times 6}{\frac{1}{2}(12 + 2\sqrt{36+36})} \\ &= \frac{6}{1 + \sqrt{2}} = 6(\sqrt{2} - 1) \\ \therefore \frac{r_2}{r_1} &= 3 \end{aligned}$$

For Problems 28–30

$$9^x - a \cdot 3^x - a + 3 \leq 0$$

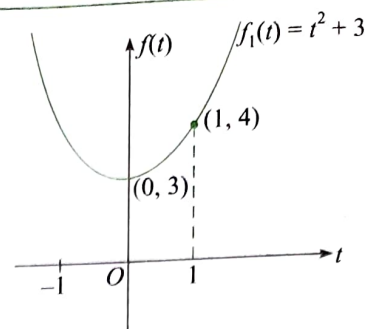
Let $t = 3^x$. Then

$$t^2 - at - a + 3 \leq 0$$

$$\text{or } t^2 + 3 \leq a(t + 1)$$

where $t \in R^+$ for $\forall x \in R$.

(1)



Let $f_1(t)$ be $t^2 + 3$ and $f_2(t)$ be $a(t + 1)$.

28. (4) For $x < 0$, $t \in (0, 1)$, i.e., (1) should have at least one solution in $t \in (0, 1)$.

From (1), it is obvious that $a \in R^+$.

Now, $f_2(t) = a(t + 1)$ represents a straight line. It should meet the curve $f_1(t) = t^2 + 3$ at least once in $t \in (0, 1)$.

$$f_1(0) = 3, f_1(1) = 4, f_2(0) = a, f_2(1) = 2a.$$

If $f_1(0) = f_2(0)$, then $a = 3$. If $f_1(1) = f_2(1)$, then $a = 2$.

Hence, $a \in (2, 3)$.

29. (3) For at least one positive solution, $t \in (1, \infty)$. Therefore, the graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$ should meet at least once in $t \in (1, \infty)$.

If $a = 2$, both the curves touch each other at $(1, 4)$.

Hence, $a \in (2, \infty)$.

30. (2) In this case, both graphs should meet at least once in $t \in (0, \infty)$.

For $a = 2$, both curves touch. Hence, $a \in [2, \infty)$.

31. (1), 32. (3)

Let the equation of line CD be

$$y = 2x + b \quad (1)$$

and the length of side of square be a

Let the coordinates of C and D be (x_1, y_1) and (x_2, y_2) , respectively. Therefore,

$$\frac{y_2 - y_1}{x_2 - x_1} = 2 \quad [\because CD \parallel AB]$$

$$\begin{aligned} \text{Also, } a^2 &= (y_2 - y_1)^2 + (x_2 - x_1)^2 \\ &= 5(x_2 - x_1)^2 \end{aligned}$$

$$\text{or } a^2 = 5[(x_1 + x_2)^2 - 4x_1x_2] \quad (2)$$

Solving (1) with $y = x^2$, we get

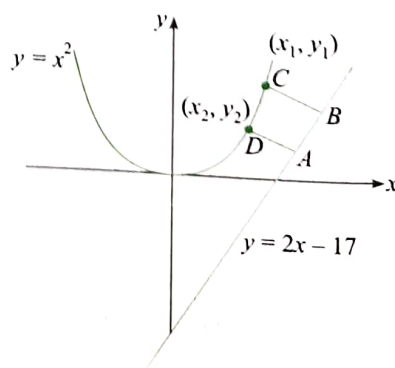
$$x^2 = 2x + b$$

$$\text{or } x_1 + x_2 = 2, x_1x_2 = -b$$

Putting in (2), we get

$$a^2 = 20(b + 1) \quad (3)$$

Now, distance CD is



$$a = \frac{2x_1 - y_1 - 17}{\sqrt{5}}$$

$$= \frac{-b-17}{\sqrt{5}}$$

or $5a^2 = (b+17)^2$ [Using (1)]

From (3) and (4), we get

$$100(b+1) = (b+17)^2 \quad (4)$$

i.e., $b = 3$ or 63

i.e., $a^2 = 80$ or $a^2 = 1280$

Matrix Match Type

1. $a \rightarrow r$; $b \rightarrow s$; $c \rightarrow p$; $d \rightarrow q$.

The locus of the point of intersection of perpendicular tangent is directrix, which is $12x - 5y + 3 = 0$.

The parabola is symmetrical about its axis, which is a line passing through the focus $(1, 2)$ and perpendicular to the directrix. So, it has equation $5x + 12y - 29 = 0$.

The minimum length of focal chord occurs along the latus rectum line, which is a line passing through the focus and parallel to the directrix, i.e., $12x - 5y - 2 = 0$.

The locus of the foot of perpendicular from the focus upon any tangent is tangent at the vertex, which is parallel to the directrix and equidistant from the directrix and latus rectum line.

Let the equation of tangent at vertex be

$$12x - 5y + \lambda = 0$$

where $\frac{|\lambda - 3|}{\sqrt{12^2 + 5^2}} = \frac{|\lambda + 2|}{\sqrt{12^2 + 5^2}}$ or $\lambda = \frac{1}{2}$

Hence, the equation of tangent at vertex is $24x - 10y + 1 = 0$.

2. $a \rightarrow q$; $b \rightarrow s$; $c \rightarrow p$; $d \rightarrow r$.

The equation of tangent having slope m is

$$y = mx + \frac{3}{m}$$

The line $3x - y + 1 = 0$ is tangent for $m = 3$.

The equation of normal having slope m is $y = mx - 6m - 3m^3$.

The line $2x - y - 36 = 0$ is normal for $m = 2$.

The chord of contact w.r.t. any point on the directrix is the focal chord which passes through the focus $(3, 0)$.

The line $2x - y - 6 = 0$ passes through the focus.

Chords which subtend right angle at the vertex are concurrent at point $(4 \times 3, 0)$, i.e., $(12, 0)$.

The line $x - 2y - 12 = 0$ passes through the point $(12, 0)$.

3. $a \rightarrow q$, s ; $b \rightarrow r$; $c \rightarrow p$, q ; $d \rightarrow q$, r .

a. Tangent to the parabola having slope m is $ty = x + t^2$. It passes through the point $(2, 3)$. Then, $3t = 2 + t^2$, i.e., $t = 1$ or 2 .
The point of contact is $(1, 2)$ or $(4, 4)$.

b. Let a point on the circle be $P(x_1, y_1)$. Then the chord of contact of the parabola w.r.t. P is $yy_1 = 2(x + x_1)$. Comparing with $y = 2(x - 2)$, we have $y_1 = 1$ and $x_1 = -2$, which also satisfy the circle.

c. Point Q on the parabola is at $(t^2, 2t)$.

Now, the area of triangle OPQ is

$$\left| \begin{vmatrix} 0 & 0 \\ 1 & 4 \\ 2 & t^2 \end{vmatrix} \right| = 6 \text{ or } 8t + 4t^2 = \pm 12$$

For $t^2 + 2t - 3 = 0$, $(t-1)(t+3) = 0$. Then $t = 1$ or $t = -3$.

Then point Q is $(1, 2)$ or $(9, -6)$.

d. Points $(1, 2)$ and $(-2, 1)$ satisfy both the curves.

4. (2)

a. We get common normal perpendicular to $y = x$.

So, slope $= -1 \Rightarrow x + y = 3a$

b. Tangent to the parabola $y = mx + a/m$ passes through the point $P(h, k)$.

$$\Rightarrow m^2h - mk + a = 0$$

If its roots are m_1 and m_2 , then $m_1m_2 = +1$.

Thus, locus is $x = a$.

c. The tangents are

$$y = m(x + a) + a/m \quad (1)$$

$$\text{and } y = \frac{-1}{m}(x + 2a) - 2am \quad (2)$$

Subtracting (1) from (2), we get

$$x + 3a = 0$$

d. If chord joining t_1 and t_1 subtends angle of 90° at vertex, then $t_1t_2 = -4$. Point of intersection of tangents is $(-at_1t_2, -a(t_1 + t_1))$. So, the locus is $x = 4a$.

Numerical Value Type

1. (7) Here, $(x-1)^2 + (y-3)^2 = \left\{ \frac{5x-12y+17}{\sqrt{5^2+(-12)^2}} \right\}^2$

Therefore, the focus is $(1, 3)$ and the directrix is $5x - 12y + 17 = 0$.

The distance of the focus from the directrix is

$$\frac{|5 \times 1 - 12 \times 3 + 17|}{\sqrt{5^2 + (-12)^2}} = \frac{14}{13}$$

$$\therefore \text{Length of latus rectum} = 2 \times \frac{14}{13} = \frac{28}{13}$$

2. (2) Points of intersection of given parabola with x -axis are $(1, 0)$ and $(4, 0)$.

Now, circle passing through these points is $(x-1)(x-4) + y^2 + \lambda y = 0$.

$$\therefore \text{Length of the tangent from } (0, 0) = \sqrt{4} = 2$$

3. (1) The focus of $y^2 = 16x$ is $(4, 0)$.

Any focal chord is

$$y - 0 = m(x - 4)$$

$$\text{or } mx - y - 4m = 0$$

This the focal chord touches the circle $(x-6)^2 + y^2 = 2$.

Then the distance from the center of the circle to this chord is equal to the radius of the circle, i.e.,

$$\frac{|6m - 4m|}{\sqrt{m^2 + 1}} = \sqrt{2}$$

$$\text{or } 2m = \sqrt{2} \cdot \sqrt{m^2 + 1}$$

$$\text{or } 2m^2 = m^2 + 1 \text{ or } m^2 = 1$$

$$\therefore m = \pm 1$$

4. (3) Any tangent to the parabola $y^2 = 4x$, ($a = 1$), is

$$y = mx + \frac{1}{m}$$

It passes through $(-2, -1)$. Therefore,

$$-1 = -2m + \frac{1}{m}$$

or $2m^2 - m - 1 = 0$

or $(2m + 1)(m - 1) = 0$

or $m = -\frac{1}{2}$ and $m = 1$

Then the angle between the lines is

$$\tan \theta = \left| \frac{m_1 + m_2}{1 - m_1 m_2} \right| = 3$$

5. (3) The equation of tangent to parabola $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

Since (1) is also the tangent of $x^2 = -32y$, we have

$$x^2 = -32 \left(mx + \frac{1}{m} \right)$$

or $x^2 + 32mx + \frac{32}{m} = 0$

The above equation must have equal roots.

Hence, its discriminant must be zero. Therefore,

$$(32m)^2 = 4 \times 1 \times \frac{32}{m}$$

i.e., $m^3 = \frac{1}{8}$ or $m = \frac{1}{2}$

From (1),

$$y = \frac{x}{2} + 2$$

or $x - 2y + 4 = 0$

6. (4) $y^2 = x$

$$\therefore 4a = 1$$

$$P(at_1^2, 2at_1) \equiv (4, -2)$$

$$\therefore t_1 = -4$$

Also, $t_1 t_2 = -1$ as PQ is a focal chord.

Slope of tangent at t_2 is $\frac{1}{t_2} = -t_1 = 4$

7. (6) Slope of the line = -1

From the curve,

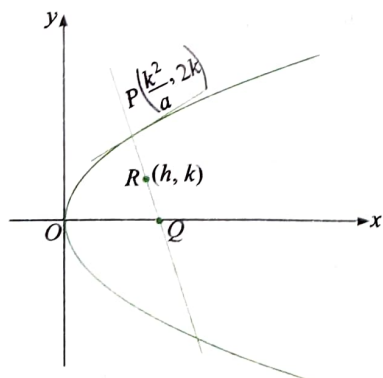
$$\frac{dy}{dx} = \frac{4}{y}$$

Hence, slope of normal = $-\frac{y}{4} = -1$ or $y = 4$

Putting $y = 4$ in the equation of curve, we have $x = 2$.

Hence, the point is (2, 4).

8. (4)



Consider the parabola $y^2 = 4ax$.

We have to find the locus of $R(h, k)$. Since Q has ordinate 0, the ordinate of P is $2k$.

Also, P is on the curve. Then the abscissa of P is k^2/a .

Now, PQ is normal to the curve.

Slope of tangent to the curve at any point is

$$\frac{dy}{dx} = \frac{2a}{y}$$

Hence, the slope of normal at point P is $-k/a$.

Also, the slope of normal joining P and $R(h, k)$ is

$$\frac{2k - k}{(k^2/a) - h}$$

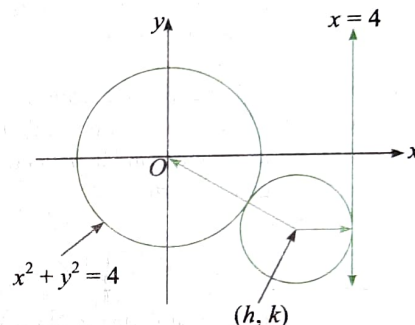
Hence, comparing slopes, we get

$$\frac{2k - k}{(k^2/a) - h} = -\frac{k}{a}$$

or $y^2 = a(x - a)$

For $y^2 = 16x$, $a = 4$. Hence, the locus is $y^2 = 4(x - 4)$.

9. (3)



The radius of variable circle is $4 - h$.

It touches $x^2 + y^2 = 4$. Therefore,

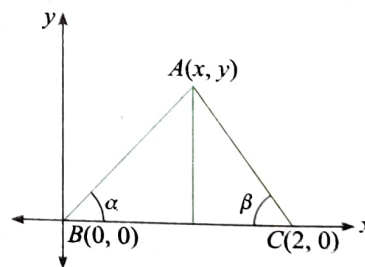
$$2 + 4 - h = \sqrt{h^2 + k^2}$$

or $x^2 + y^2 = x^2 - 12x + 36$

or $y^2 = -12(x - 3)$

The vertex is (3, 0).

10. (2.125)



Given $\tan \alpha + \tan \beta = 4$

or $\frac{y}{x} + \frac{y}{2-x} = 4$

or $y = 2x(2-x)$

or $-\frac{y}{2} = x^2 - 2x = (x-1)^2 - 1$

or $(x-1)^2 = -\frac{1}{2}(y-2)$

Directrix: $y - 2 = \frac{1}{8}$ or $y = \frac{17}{8}$

11. (8) The length of focal chord having one extremity $(at^2, 2at)$ is

$$a \left(t + \frac{1}{t} \right)^2$$

$$\therefore \left| r + \frac{1}{r} \right| \geq 2$$

$$\text{we get } \left(1 + \frac{1}{r} \right)^2 \geq 4a = 8$$

or Length of focal chord 8

$$12. (12) a = 3$$

Comparing point $(3, 6)$ with $(3r^2, 6r)$, we have $r = 1$. Then

$$\text{Length of focal chord} = a \left(r + \frac{1}{r} \right)^2 = 3(1+1)^2 = 12$$

$$13. (8) \text{ The chord of contact w.r.t. point } O(-1, 2) \text{ is}$$

$$y = (x - 1) \quad [\text{Using } yy_1 = 2a(x + x_1)]$$

Solving $y = x - 1$ with the parabola, we get the points of intersection as

$$P(3 + 2\sqrt{2}, 2 + 2\sqrt{2}) \text{ and } Q(3 - 2\sqrt{2}, 2 - 2\sqrt{2})$$

$$\therefore PQ^2 = 32 + 32 = 64$$

$$\therefore PQ = 8$$

Also, the length of perpendicular from $O(-1, 2)$ on PQ is $4/\sqrt{2}$.

Then the required area of triangle is

$$A = \frac{1}{2} \times 8 \times \left(\frac{4}{\sqrt{2}} \right) = 8\sqrt{2} \text{ sq. units}$$

$$14. (7) \text{ The line is } y = 2x - b, \text{ i.e.,}$$

$$1 = \frac{2x - y}{b}$$

Homogenizing parabola with line, we get

$$x^2 - 4x \left(\frac{2x - y}{b} \right) - y \left(\frac{2x - y}{b} \right) = 0$$

Since $\angle AOB = 90^\circ$,

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\text{or } 1 - \frac{8}{b} + \frac{1}{b} = 0$$

$$\text{or } b = 7$$

$$15. (5) \text{ Let the line be}$$

$$y = mx$$

Solving it with $5y = 2x^2 - 9x + 10$, we get

$$5mx = 2x^2 - 9x + 10$$

$$\text{or } 2x^2 - (9 + 5m)x + 10 = 0$$

$$\text{Sum of roots} = \frac{9 + 5m}{2} = 17$$

$$\text{or } 9 + 5m = 34$$

$$\text{or } 5m = 25$$

$$\text{or } m = 5$$

$$16. (5) \text{ For maximum number of common chords, the circle and the parabola must intersect at four distinct points.}$$

Let us first find the value of r when the circle and the parabola touch each other.

For that, solving the given curves, we have

$$(x - 6)^2 + 4x = r^2$$

$$\text{or } x^2 - 8x + 36 - r^2 = 0$$

The curves touch if discriminant

$$D = 64 - 4(36 - r^2) = 0$$

$$\text{or } r^2 = 20$$

Hence, the least integral value of r for which the curves intersect is 5.

$$17. (0) \text{ The family of lines } (1 + \lambda)x + (\lambda - 1)y + 2(1 - \lambda) = 0 \text{ is concurrent at } (0, 2). \text{ The focus of the parabola } x^2 = 4(y - 1) \text{ is } (0, 2).$$

Now, the shortest intercept of the line on the parabola which passes through the focus is latus rectum.

The axis of the given parabola is the y -axis.

Therefore, the latus rectum is parallel to the x -axis.

$$18. (3) \text{ Given that equation of tangent at vertex is } 3x + 4y + k = 0. \text{ Let the equation of axis be } 4x - 3y + \lambda = 0.$$

Therefore, equation of parabola is

$$(4x - 3y + \lambda)^2 = \mu(3x + 4y + k) \quad (1)$$

Given equation of parabola is

$$16x^2 - 24xy + 9y^2 + 14x + 2y + 7 = 0 \quad (2)$$

Comparing equations (1) and (2), we get

$$8\lambda - 3\mu = 14, -6\lambda - 4\mu = 2, \lambda^2 - k\mu = 7$$

Solving, we get

$$\lambda = 1, \mu = -2 \text{ and } k = 3.$$

$$19. (4) \text{ Equation of normal to parabola } y^2 = 4x \text{ having slope } m \text{ is}$$

$$y = mx - 2m - m^3$$

This normal passes through the point $P(h, k)$.

$$\therefore k = mh - 2m - m^3$$

$$\text{or } m^3 + (2 - h)m + k = 0 \quad (1)$$

This equation has three real roots m_1, m_2 and m_3 ; which are slopes of three normals.

Also, three feet of normals on the plane are $(m_1^2, -2m_1), (m_2^2, -2m_2)$ and $(m_3^2, -2m_3)$.

Given that

$$y_1y_2 + y_2y_3 + y_3y_1 = x_1x_2x_3$$

$$\therefore 4m_1m_2 + 4m_2m_3 + 4m_3m_1 = m_1^2m_2^2m_3^2 \quad (2)$$

From equation (1), we have

$$m_1m_2 + m_2m_3 + m_3m_1 = 2 - h$$

$$\text{and } m_1m_2m_3 = -k$$

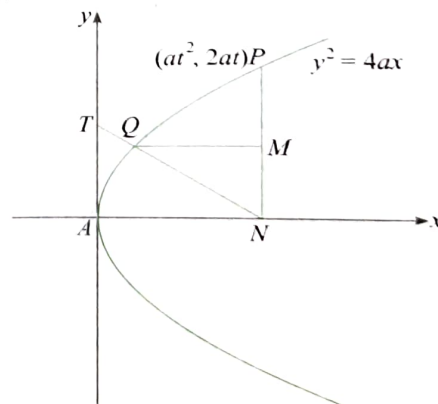
Putting these values in (2), we get

$$4(2 - h) = (-k)^2$$

$$\therefore y^2 = -4(x - 2), \text{ which is locus of point } P.$$

This equation shows a parabola with latus rectum of length 4 units.

$$20. (1.5)$$



Let point P on the parabola be $(at^2, 2at)$.

$$\therefore PN = 2at \text{ and } N \equiv (at^2, 0)$$

M is midpoint of PN .

Therefore, equation of QM is $y = at$.

Solving this with parabola, we have

$$a^2t^2 = 4ax \Rightarrow x = \frac{1}{4}at^2$$

So, $Q \equiv \left(\frac{1}{4}at^2, at \right)$

Thus, equation of NQ is

$$y - 0 = \frac{at - 0}{\frac{1}{4}at^2 - at^2}(x - at^2)$$

$$y = -\frac{4}{3t}(x - at^2)$$

If NQ meets y -axis at $T\left(0, \frac{4at}{3}\right)$, then

$$AT = \frac{4at}{3}$$

$$\therefore \frac{PN}{AT} = \frac{3}{2}$$

21. (2) Let the three points on the parabola $y^2 = 4ax$ be $A(t_1)$, $B(t_2)$ and $C(t_3)$.

Area of triangle ABC ,

$$\Delta_1 = \frac{1}{2} \begin{vmatrix} 1 & at_1^2 & 2at_1 \\ 1 & at_2^2 & 2at_2 \\ 1 & at_3^2 & 2at_3 \end{vmatrix}$$

$$= a^2 \begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix}$$

$$= |a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)| \quad (1)$$

Points of intersection of tangents at A , B and C are

$$P(at_1t_2, a(t_1 + t_2));$$

$$Q(at_2t_3, a(t_2 + t_3));$$

$$R(at_3t_1, a(t_3 + t_1)).$$

Area of triangle PQR ,

$$\Delta_2 = \frac{1}{2} \begin{vmatrix} 1 & at_1t_2 & a(t_1 + t_2) \\ 1 & at_2t_3 & a(t_2 + t_3) \\ 1 & at_3t_1 & a(t_3 + t_1) \end{vmatrix}$$

$$= \frac{a^2}{2} \begin{vmatrix} 1 & t_1t_2 & t_1 + t_2 \\ 1 & t_2t_3 & t_2 + t_3 \\ 1 & t_3t_1 & t_3 + t_1 \end{vmatrix}$$

$$= \frac{a^2}{2} \begin{vmatrix} 0 & (t_1 - t_3)t_2 & t_1 - t_3 \\ 0 & (t_2 - t_1)t_3 & t_2 - t_1 \\ 1 & t_3t_1 & t_3 + t_1 \end{vmatrix}$$

(Operating $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$)

$$= \frac{a^2}{2} |(t_1 - t_3)(t_2 - t_1)(t_2 - t_3)| \text{ (expanding w.r.t. } C_1) \quad (2)$$

From (1) and (2),

$$\frac{\Delta_1}{\Delta_2} = 2$$

22. (2) The equation of normal to the parabola $y^2 = 4x$ having slope m is

$$y = mx - 2m - m^3$$

It passes through the point $P(h, k)$. So,

$$mh - k - 2m - m^3 = 0$$

or $m^3 + (2 - h)m + k = 0 \quad (1)$

which is cubic in m and has three roots such that the product of roots

$$m_1 m_2 m_3 = -k$$

[From (1)]

But given that $m_1 m_2 = \alpha$. So,

$$m_3 = -\frac{k}{\alpha}$$

But m_3 must satisfy (1). So,

$$\frac{-k^3}{\alpha^3} + (2 - h)\left(-\frac{k}{\alpha}\right) + k = 0$$

$$\text{or } k^2 + 2\alpha^2 - h\alpha^2 - \alpha^3 = 0$$

So, the locus of $P(h, k)$ is $y^2 = \alpha^2 x + (\alpha^3 - 2\alpha^2)$.

But given that the locus of P is a part of the parabola $y^2 = 4x$.

Therefore, comparing the two, we get

$$\alpha^2 = 4 \text{ and } \alpha^3 - 2\alpha^2 = 0$$

$$\therefore \alpha = 2$$

Archives

JEE Main

Single Correct Answer Type

- (4) The locus of the perpendicular tangents is the directrix; hence, the locus of P is $x = -1$.
- (2) Tangent to parabola $y^2 = 4\sqrt{5}x$, having slope m is:

$$y = mx + \frac{\sqrt{5}}{m}$$

This line is tangent to the circle $2x^2 + 2y^2 = 5$

$\Rightarrow$ Distance of center of circle from the line is equal to its radius.

$$\Rightarrow \left| \frac{m \times 0 - 0 + \frac{\sqrt{5}}{m}}{\sqrt{1 + m^2}} \right| = \sqrt{\frac{5}{2}}$$

$$\Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m^2 = 1 \Rightarrow m = \pm 1$$

Also $m^2 = 1$ satisfies $m^4 - 3m^2 + 2 = 0$

One of the common tangents for $m = 1$ is $y = x + \sqrt{5}$.

Thus, both statements are correct, but statement 2 is not correct explanation of statement 1.

- (1) Equation of tangent to $y^2 = 4x$ at $A(t^2, 2t)$ is

$$yt = x + t^2$$

This is tangent to $x^2 + 32 = 0$

$$\Rightarrow x^2 + 32\left(\frac{x}{t} + t\right) = 0 \Rightarrow x^2 + \frac{32}{t}x + 32t = 0$$

Above equation must have equal roots

$$\Rightarrow \left(\frac{32}{t}\right)^2 - 4(32t) = 0$$

$$\Rightarrow 32\left(\frac{32}{t^2} - 4t\right) = 0$$

$$\Rightarrow t^3 = 8 \Rightarrow t = 2$$

$$\Rightarrow \text{Slope of tangent is } \frac{1}{t} = \frac{1}{2}$$

4. (4) Let P be (h, k) .

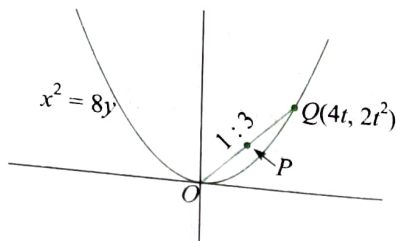
$$\therefore h = t$$

$$\text{and } k = \frac{t^2}{2}$$

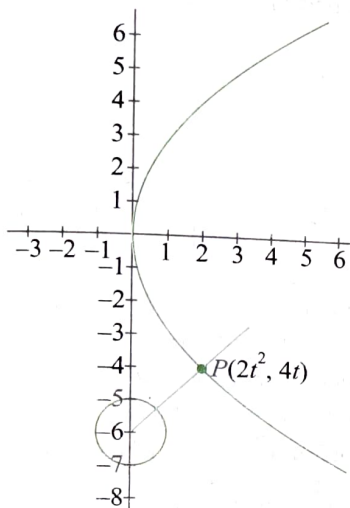
$$\therefore k = \frac{h^2}{2}$$

$$\Rightarrow 2y = x^2,$$

which is required locus.



5. (4) Point P lies on the common normal to parabola at point P and circle.



For $y^2 = 8x$, point P is $(2t^2, 4t)$

Equation of normal at this point P is $y = -tx + 4t + 2t^3$.

This is also normal to the given circle, so it must pass through the center of the circle.

$$\therefore -6 = 4t + 2t^3$$

$$\therefore t^3 + 2t + 3 = 0$$

$$\therefore (t+1)(t^2 - t + 3) = 0$$

$$\therefore t = -1$$

$$\therefore \text{Point } P \text{ is } (2, -4)$$

$$\therefore \text{Equation of required circle is } (x-2)^2 + (y+4)^2 = r^2 = 8$$

$$\text{Or } x^2 + y^2 - 4x + 8y + 12 = 0$$

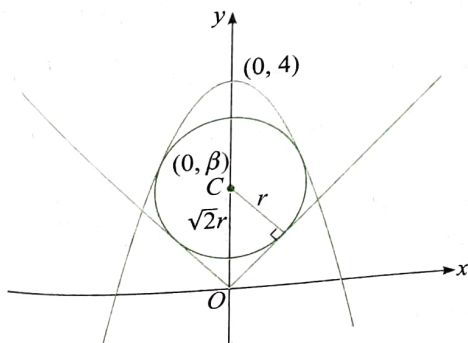
6. (None)

Let the centre be $(0, \beta)$.

Length of perpendicular from $(0, \beta)$ to $y = x$,

$$\text{i.e., } r = \left| \frac{\beta}{\sqrt{2}} \right|.$$

$$\text{So, equation of circle is } x^2 + (y - \beta)^2 = \frac{\beta^2}{2}.$$



Solving circle and parabola, we get

$$4 - y + y^2 - 2\beta y + \frac{\beta^2}{2} = 0$$

$$\text{or } y^2 - (2\beta + 1)y + \left(\frac{\beta^2}{2} + 4\right) = 0$$

Since circle touches the parabola, above equation has equal roots.

$$\therefore D = 0$$

$$\Rightarrow (2\beta + 1)^2 - 4\left(\frac{\beta^2}{2} + 4\right) = 0$$

$$\Rightarrow 2\beta^2 + 4\beta - 15 = 0$$

$$\Rightarrow \beta = \frac{-2 + \sqrt{34}}{2}$$

$$\text{Therefore radius} = \frac{\beta}{\sqrt{2}} = \frac{-2 + \sqrt{34}}{2\sqrt{2}}$$

None of the options is correct.

7. (1) Differentiating curve $x^2 = y - 6$ w.r.t. x , we get

$$\frac{dy}{dx} = 2x$$

Therefore, slope of tangent to the curve at point $P(1, 7)$ is

$$\left(\frac{dy}{dx}\right)_{(1,7)} = 2$$

Equation of tangent to the curve at point P is

$$y - 7 = 2(x - 1)$$

$$\text{or } 2x - y + 5 = 0$$

(1)

Given circle is $x^2 + y^2 + 16x + 12y + c = 0$.

Center of the circle is $C(-8, -6)$.

$$\text{Radius of the circle, } r = \sqrt{64 + 36 - c}$$

Line (1) touches the circle at point M .

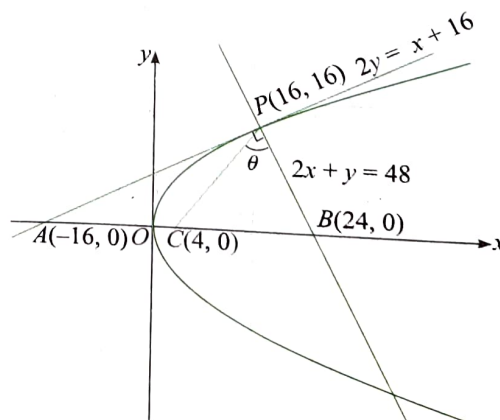
$$\text{So, } CM = r$$

$$\frac{|-16 + 6 + 5|}{\sqrt{2^2 + (-1)^2}} = \sqrt{64 + 36 - c}$$

$$\therefore \sqrt{100 - c} = \sqrt{5}$$

$$\therefore c = 95$$

8. (3) Given parabola is $y^2 = 16x$.



Equation of tangent at point $P(16, 16)$ is

$$16y = 8(x + 16) \text{ or } 2y = x + 16$$

It meets x -axis at $A(-16, 0)$.

Slope of normal is -2 .

Equation of normal at P is

$$y - 16 = -2(x - 16) \text{ or } 2x + y = 48$$

It meets x -axis at $B(24, 0)$.

Circumcircle of triangle APB has center C at mid-point of AB .

$$\therefore C \equiv (4, 0)$$

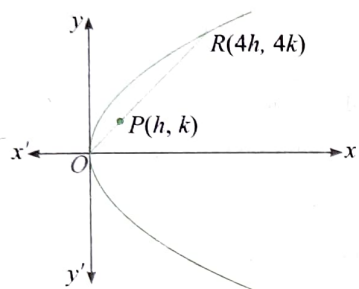
$$\text{Slope of } CP = \frac{16}{12} = \frac{4}{3}$$

$$\therefore \tan \theta = \left| \frac{\frac{4}{3} - (-2)}{1 + \frac{4}{3}(-2)} \right| = \left| \frac{10}{-5} \right| = 2$$

EE Advanced

Single Correct Answer Type

1. (3)



Point $P(h, k)$ divides the line segment OR in ratio $1 : 3$.

So, coordinates of point R are $(4h, 4k)$.

(Using ratio $OP : PR = 1 : 3$)

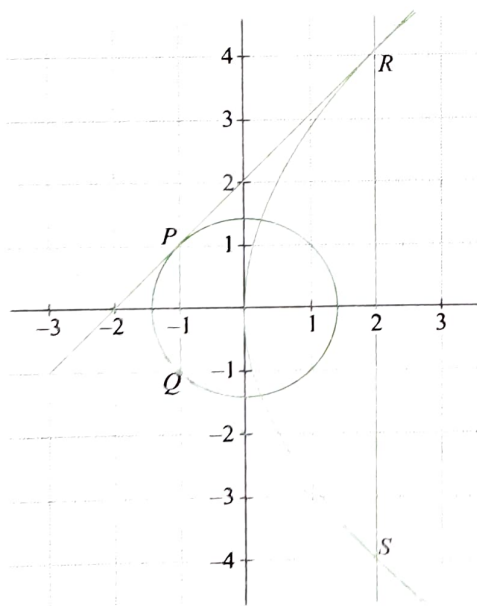
Point R lies on the parabola

$$\therefore (4k)^2 = 4 \times 4h \text{ or } k^2 = h$$

Hence, locus is $y^2 = x$.

2. (4) Equation of tangent to parabola $y^2 = 8x$ having slope m is

$$y = mx + \frac{2}{m}$$



It is also tangent to circle $x^2 + y^2 = 2$

$$\Rightarrow \frac{\frac{2}{m}}{\sqrt{1+m^2}} = \pm \sqrt{2}$$

$$\Rightarrow m = \pm 1$$

So, equation of tangents are $y = \pm(x + 2)$.

Chord of contact $PQ : (-2)x + y(0) = 2$ or $x = -1$

Chord of contact $RS : y \times 0 = 4(x - 2)$ or $x = 2$

Line $x = -1$ meets the circle at points $P(-1, 1)$ and $Q(-1, -1)$.

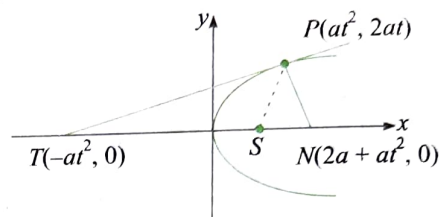
Line $x = 2$ meets the parabola at points $R(2, 4)$ and $S(2, -4)$.

Area of trapezium $PQRS$

$$= \frac{1}{2}(PQ + RS) \times (\text{Height}) = \frac{1}{2}(10) \times (3) = 15 \text{ sq. units}$$

Multiple Correct Answers Type

1. (1), (4)



Tangent at point $P(at^2, 2at)$ is $ty = x + at^2$.

It meets the x -axis at $(-at^2, 0)$.

Normal at point P is $y = -tx + 2at + at^3$.

It meets the x -axis at $(2a + at^2, 0)$.

Let the centroid of triangle PNT be $G \equiv (h, k)$. Then,

$$h = \frac{2a + at^2}{3} \text{ and } k = \frac{2at}{3}$$

Eliminating t , we get

$$\therefore \left(\frac{3h - 2a}{a} \right) = \frac{9k^2}{4a^2}$$

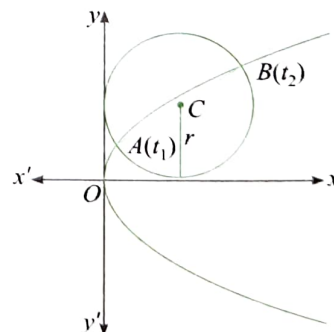
So, the required parabola is

$$\frac{9y^2}{4a^2} = \frac{(3x - 2a)}{a} = \frac{3}{a} \left(x - \frac{2a}{3} \right)$$

$$\text{or } y^2 = \frac{4a}{3} \left(x - \frac{2a}{3} \right)$$

$$\text{Vertex} \equiv \left(\frac{2a}{3}, 0 \right); \text{focus} \equiv \left(\frac{2a}{3} + \frac{a}{3}, 0 \right) \equiv (a, 0)$$

2. (3), (4)



We have points $A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$ on the parabola $y^2 = 4x$.

For circle on AB as diameter center is $C \left(\frac{t_1^2 + t_2^2}{2}, (t_1 + t_2) \right)$.

Since circle is touching the x -axis, we have $r = |t_1 + t_2|$

$$\text{or } t_1 + t_2 = \pm r$$

$$\text{Also slope of } AB, m = \frac{2t_1 - 2t_2}{t_1^2 - t_2^2} = \frac{2}{t_1 + t_2} = \pm \frac{2}{r}$$

3. (1), (2), (4)

$$y^2 = 4x$$

The equation of normal is $y = mx - 2m - m^3$.

It passes through (9, 6). So,

$$m^3 - 7m + 6 = 0$$

or $(m-1)(m-2)(m+3) = 0$

or $m = 1, 2, -3$

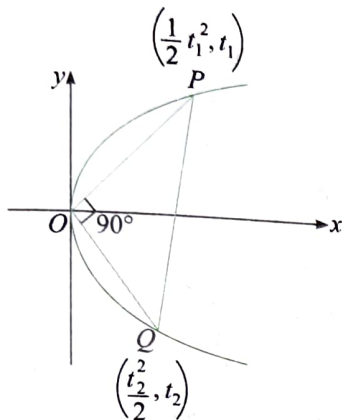
$\therefore$ normals are $y - x + 3 = 0, y + 3x - 33 = 0, y - 2x + 12 = 0$

4. (1), (4)

$$OP \perp OQ$$

$$\Rightarrow t_1 t_2 = -4$$

$$\Rightarrow \text{Area of } \Delta OPQ = \frac{1}{2} OP \cdot OQ$$



$$\Rightarrow \left| \frac{1}{2} \sqrt{\frac{t_1^4}{4} + t_1^2} \sqrt{\frac{t_2^4}{4} + t_2^2} \right| = 3\sqrt{2}$$

$$\Rightarrow \frac{1}{2} \cdot 4 \cdot \sqrt{\frac{(t_1^2 + 4)}{4} \frac{(t_2^2 + 4)}{4}} = 3\sqrt{2}$$

$$\Rightarrow 4 \cdot \frac{(16 + 4(t_1^2 + t_2^2) + 16)}{16} = 9 \times 2$$

$$\Rightarrow 8 + t_1^2 + t_2^2 = 18$$

$$\Rightarrow t_1^2 + t_2^2 - 10 = 0$$

$$\Rightarrow t_1^4 - 10t_1^2 + 16 = 0$$

$$\Rightarrow t_1^2 = 2, 8$$

5. (1), (3), (4)

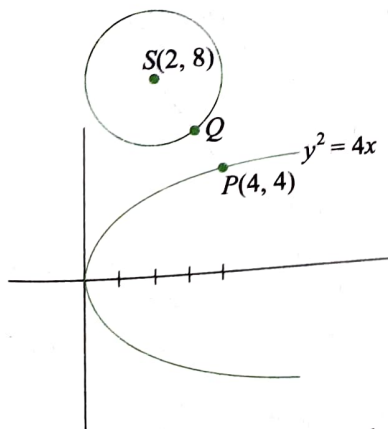
Given circle is:

$$x^2 + y^2 - 4x - 16y + 64 = 0$$

Or $(x-2)^2 + (y-8)^2 = 4$

Centre $S \equiv (2, 8)$

Normal to parabola is $y = mx - 2m - m^3$



For shortest distance from the centre, normal must pass through the centre of the circle.

$$\Rightarrow 8 = 2m - 2m - m^3$$

$$\Rightarrow m = -2$$

$$\Rightarrow \text{Normal at } P, y = -2x + 12$$

$$\therefore \text{Point } P \equiv (am^2, -2am) \equiv (4, 4)$$

$$\therefore SP = \sqrt{(4-2)^2 + (8-4)^2} = 2\sqrt{5}$$

$$\therefore SQ : QP = 2 : (2\sqrt{5} - 2)$$

$$\text{Slope of tangent at } Q = \frac{1}{2} \quad (\text{Tangent at } Q \text{ is perpendicular to } SP)$$

6. (1), (2), (3)

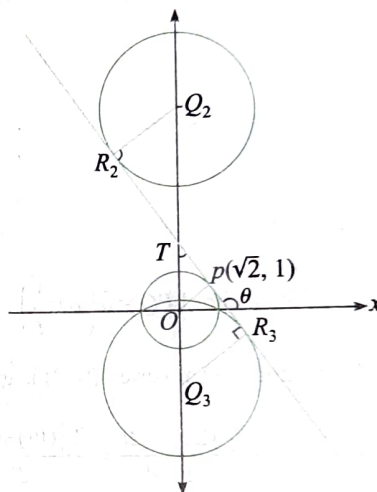
$$x^2 + y^2 = 3 \text{ and } x^2 = 2y$$

Solving above curves, we get point of intersection as $P \equiv (\sqrt{2}, 1)$.

Equation of tangent to circle at this point is $\sqrt{2}x + y = 3$.

$$\therefore \tan \theta = -\sqrt{2}$$

$$\tan \alpha = \tan(\theta - 90^\circ) = -\cot \theta = \frac{1}{\sqrt{2}}$$



$$\text{So, } \sin \alpha = \frac{1}{\sqrt{3}} = \frac{Q_3 R_3}{Q_3 T} = \frac{2\sqrt{3}}{Q_3 T}$$

$$\Rightarrow Q_3 T = 6$$

$$\therefore Q_2 Q_3 = 2 Q_3 T = 12 \quad (\text{As circles have equal radii})$$

$$\tan \alpha = \frac{1}{\sqrt{2}} = \frac{Q_3 R_3}{R_3 T} = \frac{2\sqrt{3}}{R_3 T}$$

$$\Rightarrow R_3 T = 2\sqrt{6}$$

$$\Rightarrow R_2 R_3 = 2 R_3 T = 4\sqrt{6}$$

$$\text{Perpendicular distance of } O \text{ from } R_2 R_3 = \left| \frac{3}{\sqrt{(\sqrt{2})^2 + 1^2}} \right| = \sqrt{3}$$

$$\therefore \text{Area } (OR_2 R_3) = \frac{1}{2} \times \sqrt{3} \times 4\sqrt{6} = 6\sqrt{2} \text{ sq. units}$$

$$\text{Similarly, Area } (PQ_2 Q_3) = \frac{1}{2} \times \sqrt{2} \times 12 = 6\sqrt{2} \text{ sq. units}$$

7. (4) Parabola, $y^2 = 16x$

$$\text{Equation of chord, } 2x + y = p$$

$$\text{Equation of chord having mid point } (h, k) \text{ is } T = S_1$$

$$ky - 8(x + h) = k^2 - 16h$$

$$\text{or } ky - 8x = k^2 - 8h$$

(1)

(2)

Comparing (1) and (2), we get

$$\frac{k}{1} = \frac{-8}{2} = \frac{k^2 - 8h}{p}$$

$$\Rightarrow k = -4 \text{ and } k^2 - 8h = -4p$$

From this, we get

$$16 - 8h = -4p$$

Clearly, $h = 3$ and $p = 2$.

Linked Comprehension Type

1. (2) Since PQ is focal chord, let $P \equiv (at^2, 2at)$ and $Q \equiv (a/t^2, -2a/t)$.

We know that point of intersection of tangents at $P(t_1)$ and $Q(t_2)$ is $(at_1 t_2, a(t_1 + t_2))$

$\therefore$ The point of intersection of tangents at P and Q is

$$\left(-a, a\left(t - \frac{1}{t}\right)\right)$$

As the point of intersection lies on $y = 2x + a$, we have

$$a\left(t - \frac{1}{t}\right) = -2a + a$$

$$\text{or } t - \frac{1}{t} = -1$$

$$\text{or } \left(t + \frac{1}{t}\right)^2 = 5$$

$$\therefore \text{Length of focal chord } PQ = a\left(t + \frac{1}{t}\right)^2 = 5a$$

2. (4) Angle made by chord PQ at vertex (0, 0) is given by

$$\tan \theta = \left| \frac{m_{OP} - m_{OQ}}{1 + m_{OP} \cdot m_{OQ}} \right| = \left| \frac{(2/t) + 2t}{1 - 4} \right| = \frac{2\{(1/t) + t\}}{-3} = \frac{-2\sqrt{5}}{3}$$

3. (4) Slope (PK) = Slope (QR)

$$\frac{2at - 0}{at^2 - 2a} = \frac{-2a - 2ar}{\frac{a}{t^2} - ar^2}$$

$$\Rightarrow \frac{t}{t^2 - 2} = -\left(\frac{\frac{1}{t} + r}{\frac{1}{t^2} - r^2}\right)$$

$$\Rightarrow \frac{1}{t^2 - 2} = \frac{1 + rt}{t^2 r^2 - 1}$$

$$\Rightarrow t^2 r^2 - 1 = t^2 + rt^3 - 2 - 2rt$$

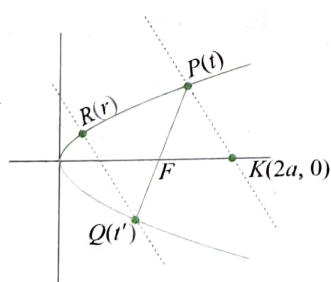
$$\Rightarrow t^2 r^2 + (2 - t^2)rt + (1 - t^2) = 0$$

$$\Rightarrow tr = -1 \text{ or } tr = t^2 - 1$$

$$\Rightarrow r = -1/t \text{ or } r = \frac{t^2 - 1}{t}$$

But for $r = -1/t$, points Q and R are coincident.

$$\therefore r = \frac{t^2 - 1}{t}$$



4. (2) Tangent at $P: ty = x + at^2$ or $y = \frac{x}{t} + at$

$$\text{Normal at } S: y + \frac{x}{t} = \frac{2a}{t} + \frac{a}{t^3}$$

$$\text{Solving, } 2y = at + \frac{2a}{t} + \frac{a}{t^3}$$

$$\Rightarrow y = \frac{a(t^2 + 1)^2}{2t^3}$$

Matrix Match Type

1. $a \rightarrow s$; $b \rightarrow p$; $c \rightarrow q$; $d \rightarrow r$.

Let point F on the parabola be $(4t^2, 8t)$.

Tangent at this point is $ty = x + 4t^2$.

It meets the y-axis at $(0, 4t)$.

Then the area of triangle EFG is $A(t) = 2t^2(3 - 4t) = 6t^2 - 8t^3$.

Differentiating w.r.t. t , we get

$$A'(t) = 12t - 24t^2$$

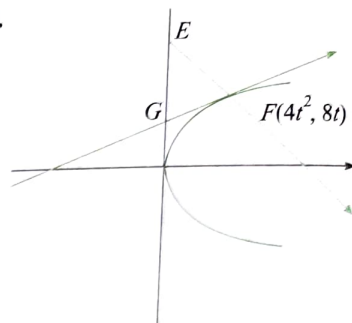
For $A'(t) = 0$, $t = 1/2$, which is a point of maxima. So, point F is $(1, 4)$.

Slope of EF = 1

$$\therefore m = 1 \text{ or } A(t)|_{\max} = \frac{1}{2} \text{ sq. units}$$

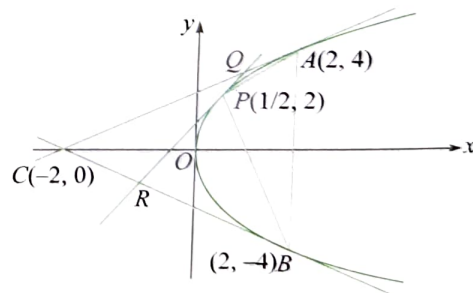
$$y_0 = 4$$

$$\text{and } y_1 = 2$$



Numerical Value Type

1. (2)



$$y^2 = 8x$$

Tangents at the end points of latus rectum meet the directrix on the x-axis at $(-2, 0)$.

$$\Delta_1 = \text{Area of } \triangle ABP$$

$$= \frac{1}{2}(8)\left(2 - \frac{1}{2}\right) = 6$$

$$\Delta_2 = \text{Area of } \triangle CQR$$

Now, tangent at point $A(2, 4)$ is $4y = 4(x + 2)$ or $y = x + 2$.

Tangent at point $B(2, -4)$ is $-4y = 4(x + 2)$ or $y = -x - 2$.

Also, tangent at point $P(1/2, 2)$ is $2y = 4(x + 1/2)$

$$\text{or } y = 2x + 1$$

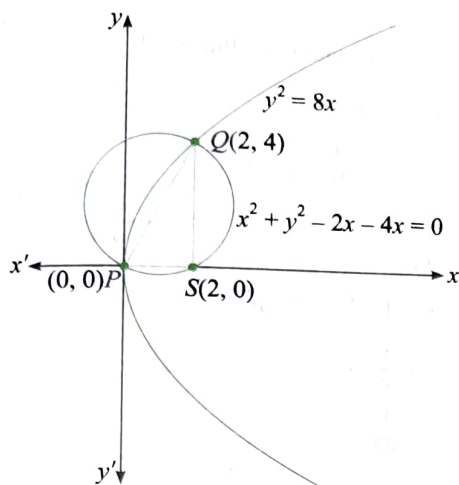
Solving for Q and R, we get $Q(1, 3)$ and $R(-1, -1)$.

Hence, area of $\triangle CQR$ is

$$\frac{1}{2} \begin{vmatrix} -1 & -1 & 1 \\ 1 & 3 & 1 \\ -2 & 0 & 1 \end{vmatrix} = \frac{1}{2}(-3+2+6+1)=3$$

Hence, the ratio of area is $6/3 = 2$.

2. (4)



We have parabola $y^2 = 8x$ with circle $x^2 + y^2 - 2x - 4y = 0$

Solving parabola $(2t^2, 4t)$ with circle, we get

$$4t^4 + 16t^2 - 4t^2 - 16t = 0$$

$$\text{or } t^4 + 3t^2 - 4t = 0$$

$$\Rightarrow t = 0, 1$$

So, the points P and Q are $(0, 0)$ and $(2, 4)$, respectively which are also diametrically opposite points on the circle. The focus is $S(2, 0)$.

$$\text{Area of } \triangle PQS = \frac{1}{2} \times 2 \times 4 = 4 \text{ sq. units}$$

3. (4) Let $P(t^2, 2t)$ be a point on the curve $y^2 = 4x$ and $Q(h, k)$ be it's image in $x + y + 4 = 0$.

$$\therefore \frac{h - t^2}{1} = \frac{k - 2t}{1} = -\frac{2(t^2 + 2t + 4)}{2}$$

$$\Rightarrow h = -(2t + 4) \text{ and } k = -(t^2 + 4)$$

Now $y = -5$ intersect this locus.

$$\therefore k = -5, \text{ so } t = \pm 1$$

$$\text{Hence, } h = -2, -6$$

So, points of intersection are $A(-2, -5)$ and $B(-6, -5)$.

$$\text{Hence, } AB = 4.$$

4. (2) End points of latus rectum of parabola $y^2 = 4x$ are $(1, \pm 2)$.

Equation of normals at points $(1, \pm 2)$ are

$$y = -x + 3 \text{ and } y = x - 3$$

$$\text{or } x + y - 3 = 0 \text{ and } x - y - 3 = 0$$

These lines are tangent to circle. $(x - 3)^2 + (y + 2)^2 = r^2$

$$\therefore \left| \frac{3 \pm 2 - 3}{\sqrt{1+1}} \right| = r$$

$$\text{or } r^2 = 2$$

Chapter 6

Concept Application Exercises

Exercise 6.1

1. Equation of ellipse is locus of point $P(x, y)$ such that $SP = e \cdot PM$, where $S \equiv (1, 2)$ and M is foot of perpendicular from focus upon directrix.

Therefore, the equation of ellipse is

$$\sqrt{(x-1)^2 + (y-2)^2} = \frac{1}{2} \frac{|x-y+2|}{\sqrt{1^2 + (-1)^2}}$$

$$\text{or } 8(x^2 - 2x + 1 + y^2 - 4y + 4) = x^2 + y^2 + 4 + 4x - 4y - 2xy$$

$$\text{or } 7x^2 + 7y^2 + 2xy - 20x - 28y + 36 = 0$$

2. We have $\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + (y-\sqrt{12})^2} = a$
distance between $P(x, y)$ and $S(1, 0)$ distance between $P(x, y)$ and $S'(-1, \sqrt{12})$

$$\therefore SP + S'P = a$$

So, locus of P is ellipse if

$$a > SS'$$

$$\text{or } a > \sqrt{4+12}$$

$$\text{or } a > 4$$

3. $(3x-12)^2 + (3y+15)^2 = \frac{(3x+4y+5)^2}{25}$

$$\text{or } \sqrt{(x-4)^2 + (y+5)^2} = \frac{1}{3} \frac{|3x-4y+5|}{5}$$

Here, the ratio of distance of the variable point $P(x, y)$ from the fixed point (focus) $(4, 5)$ to that from the fixed line (directrix) $3x-4y+5=0$ is $1/3$.

Hence, the locus is an ellipse and its eccentricity is $1/3$.

Also, length of latus rectum

$$= 2(e) \times (\text{Distance of } (4, 5) \text{ from the line } 3x-4y+5=0)$$

$$= 2 \times \frac{1}{3} \frac{|3 \times 4 - 4 \times 5 + 5|}{5} = \frac{2}{5}$$

Exercise 6.2

1. Let equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Given that, $e = \frac{2}{3}$ and latus rectum = 5.

If $a > b$, then

$$\frac{2b^2}{a} = 5 \Rightarrow b^2 = \frac{5a}{2}$$

$$\text{Now, } b^2 = a^2(1-e^2)$$

$$\Rightarrow \frac{5a}{2} = a^2 \left(1 - \frac{4}{9}\right)$$

$$\Rightarrow \frac{5}{2} = \frac{5a}{9}$$

$$\therefore a = \frac{9}{2}$$

$$\therefore b^2 = \frac{5 \times 9}{2 \times 2} = \frac{45}{4}$$

So, the required equation of the ellipse is $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$.

If $a < b$, then equation of ellipse will be $\frac{4x^2}{45} + \frac{4y^2}{81} = 1$.

2. $m_{BF} \cdot m_{B'F} = -1$

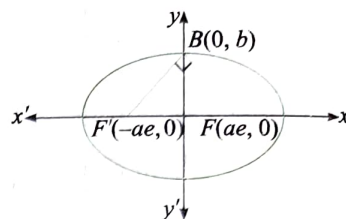
$$\text{or } \frac{b-0}{0-ae} \times \frac{b-0}{0+ae} = -1$$

$$\text{or } \frac{b^2}{a^2 e^2} = 1$$

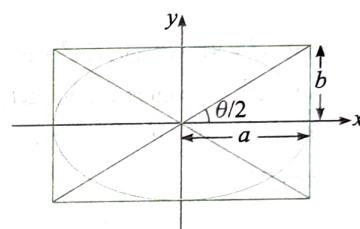
$$\text{or } e^2 = 1 - \frac{b^2}{a^2} = 1 - e^2$$

$$\text{or } e^2 = \frac{1}{2}$$

$$\text{or } e = \frac{1}{\sqrt{2}}$$



- 3.



From the figure, we have

$$\tan \frac{\theta}{2} = \frac{b}{a} \Rightarrow \tan^2 \frac{\theta}{2} = \frac{b^2}{a^2}$$

$$\Rightarrow \frac{1 - \cos \theta}{1 + \cos \theta} = \frac{b^2}{a^2}$$

$$\Rightarrow \frac{1 - \frac{1}{3}}{1 + \frac{1}{3}} = \frac{b^2}{a^2} \Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$$

4. Here, $ae = 4$ and $e = 4/5$. So, $a = 5$.

$$\text{Now, } b^2 = a^2(1-e^2)$$

$$\text{or } b^2 = 25 \left(1 - \frac{16}{25}\right) = 9$$

Hence, the equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

5. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

According to the question, $a = 3$ and $b = 2$.

$$\text{Now, } b^2 = a^2(1-e^2)$$

$$\Rightarrow e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\therefore e = \frac{\sqrt{5}}{3}$$

From the definition of the ellipse for any point P on the ellipse,

$$SP + S'P = 2a, \text{ where } S \text{ and } S' \text{ are foci.}$$

Length of the endless string = $SP + S'P + SS'$

$$= 2a + 2ae$$

$$= 2 \times 3 + 2 \times 3 \times \frac{\sqrt{5}}{3}$$

$$= 6 + 2\sqrt{5}$$

6. Let the equation of the semi-elliptical arc be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (y > 0)$$

Length of the major axis = $2a = 9$

$$\text{or } a = \frac{9}{2}$$

Length of the semi-minor axis = $b = 3$

So, the equation of the arc becomes

$$\frac{4x^2}{81} + \frac{y^2}{9} = 1$$

If $x = 2$, then

$$y^2 = \frac{65}{9} \text{ or } y = \frac{1}{3} \sqrt{65} = \frac{8}{3} \text{ approximately}$$

7. Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a < b)$$

Since the foci are $(0, \pm 1)$, we have

$$be = 1 \text{ and } 2a = 1 \quad [\text{Since minor axis is of length 1}]$$

Also, $a^2 = b^2(1 - e^2)$

$$\text{or } \frac{1}{4} = b^2 - b^2 e^2 = b^2 - 1$$

$$\text{or } b^2 = 1 + \frac{1}{4} = \frac{5}{4}$$

hence, the equation of the ellipse is

$$\frac{x^2}{1/4} + \frac{y^2}{5/4} = 1 \text{ or } 20x^2 + 4y^2 = 5$$

8. For given ellipse, $a = 7$ and $b = 8$.

Let P be any point on the ellipse, then $SP + S'P = 2b = 16$.

Given $SP = 10$ then $S'P = 6$

9. Let the required ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,
where $a^2 = b^2(1 - e^2)$.

Ellipse passes through the point $(2, 4)$.

$$\therefore \frac{4}{a^2} + \frac{16}{b^2} = 1$$

Also, distance between foci is $4\sqrt{6}$.

$$\therefore be = 2\sqrt{6}$$

$$\therefore \frac{4}{b^2 - 24} + \frac{16}{b^2} = 1$$

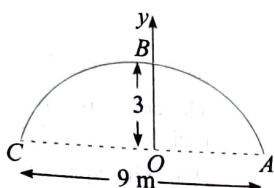
$$\Rightarrow b^4 - 44b^2 + 16 \times 24 = 0$$

$$\Rightarrow (b^2 - 32)(b^2 - 12) = 0$$

$$\therefore b^2 = 32 \text{ (as } b > 4)$$

$$\text{and } a^2 = 8$$

Therefore, the required equation is $\frac{x^2}{8} + \frac{y^2}{32} = 1$.



10. We have $2ae = 10$ or $ae = 5$

$$\text{and } \frac{a}{e} - ae = 15$$

$$\Rightarrow \frac{5}{e^2} - 5 = 15$$

$$\Rightarrow \frac{5}{e^2} = 20$$

$$\therefore e = \frac{1}{2}$$

11. In the figure, B is orthocentre.

$$\text{Slope of altitude } S'B, m_1 = \frac{b}{ae}$$

$$\text{Slope of } SA, m_2 = \frac{a}{-ae}$$

Since, $m_1 \times m_2 = -1$, we have

$$\frac{b}{ae} \times \left(\frac{1}{-e} \right) = -1$$

$$\Rightarrow 1 - e^2 = e^4$$

$$\Rightarrow e^4 + e^2 - 1 = 0$$

$$\Rightarrow e^2 = \frac{\sqrt{5} - 1}{2}$$

$$\Rightarrow e = \sqrt{\frac{\sqrt{5} - 1}{2}}$$

12. Given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$ ($a > b$).

$$a^2 = 9, b^2 = 4, e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{9 - 4}{9}} = \frac{\sqrt{5}}{3}$$

Greatest focal distance = $a + ae$

Least focal distance = $a - ae$

$$\therefore \text{Required ratio} = \frac{a + ae}{a - ae} = \frac{1 + e}{1 - e}$$

$$\begin{aligned} &= \frac{1 + \frac{\sqrt{5}}{3}}{1 - \frac{\sqrt{5}}{3}} = \frac{3 + \sqrt{5}}{3 - \sqrt{5}} \\ &= \frac{(3 + \sqrt{5})^2}{4} = \frac{7 + 3\sqrt{5}}{2} \end{aligned}$$

13. Given ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1 \quad (1)$$

$$\therefore e^2 = 1 - \frac{16}{25} \text{ or } e = \frac{3}{5}$$

Then foci are $(\pm ae, 0) \equiv (\pm 3, 0)$.

Now, the circle having center $(3, 0)$ is

$$(x - 3)^2 + y^2 = r^2 \quad (2)$$

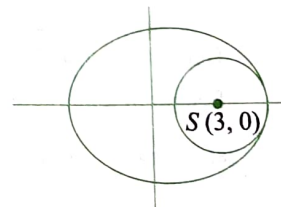
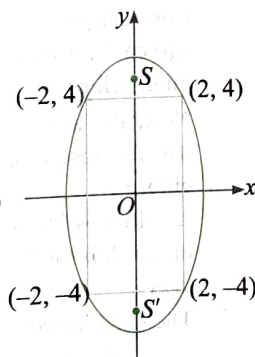
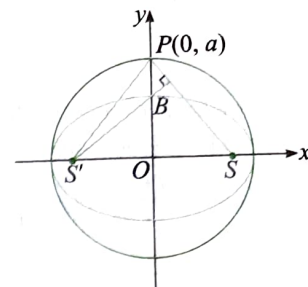
Eliminating y^2 from (1) and (2), we get

$$16x^2 + 25r^2 - 25(x^2 - 6x + 9) = 400$$

$$\text{or } -9x^2 + 150x + 25r^2 - 625 = 0$$

since the circle touches the ellipse, the above equation has equal roots. Hence, $D = 0$, i.e., $22500 + 36(25r^2 - 625) = 0$, which is not possible.

Then the circle will touch the ellipse at the end of the major axis, $(5, 0)$. Hence, the radius is 2.



14. As shown in the figure, the earth is at focus A , and the moon is travelling in an elliptical orbit.

The maximum distance from the moon to the earth is $AC = 405500$ km.

The minimum distance from the moon to the earth is $AB = 363300$ km.

Considering the equation of the orbit as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we have $a - ae = 363300$ km

and $a + ae = 405500$

$$\therefore \frac{a + ae}{a - ae} = \frac{405500}{363300}$$

$$\text{or } \frac{1 + e}{1 - e} = \frac{4055}{3633}$$

$$\text{or } e = \frac{4055 - 3633}{4055 + 3633} = \frac{422}{7688} \approx 0.05489$$

15. The given equation can be written as

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

which represents an ellipse whose center is $(-1, -2)$ and semi-major and the minor axes are 5 and 3, respectively.

The eccentricity of the ellipse is given by

$$9 = 25(1 - e^2) \text{ or } e = \frac{4}{5}$$

Shifting the origin at $(-1, -2)$, the given equation reduces to

$$\frac{X^2}{9} + \frac{Y^2}{25} = 1 \quad (1)$$

$$\text{where } x = X - 1, y = Y - 2. \quad (2)$$

The coordinates of the foci of (1) are $(X = 0, Y = \pm be)$, where $b = 5, e = 4/5$, i.e., the foci of (1) are $(X = 0, Y = \pm 4)$.

Therefore, the coordinates of the foci of the given ellipse are $(-1, 2)$ and $(-1, -6)$.

16. Area bounded by the given circle and ellipse

$$= \text{Area of circle} - \text{Area of ellipse}$$

$$= \pi a^2 - \pi ab = \pi a(a - b)$$

$$= \text{Area of ellipse having semi-axis } a - b \text{ and } b$$

17. The given equation is

$$\frac{\{(3x - 4y + 2)/5\}^2}{16/25} + \frac{\{(4x - 3y - 5)/5\}^2}{9/25} = 1$$

Hence, $a^2 = 16/25$ and $b^2 = 9/25$. Therefore,

$$\text{Length of major axis} = 2a = \frac{8}{5}$$

$$\text{and Length of minor axis} = 2b = \frac{6}{5}$$

$$e^2 = 1 - \frac{9}{16} \text{ or } e = \frac{\sqrt{7}}{4}$$

18. The equation of the chord of ellipse which gets bisected at $P(h, k)$ is

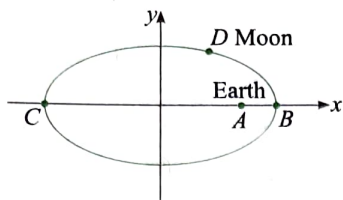
$$\frac{hx}{4} + \frac{ky}{9} = \frac{h^2}{4} + \frac{k^2}{9} \quad (1)$$

Its distance from the origin $(0, 0)$ is 2. Therefore,

$$\frac{|0 + 0 - (\frac{h^2}{4} + \frac{k^2}{9})|}{\sqrt{\frac{h^2}{16} + \frac{k^2}{81}}} = 2$$

Therefore, the locus is

$$4\left(\frac{x^2}{16} + \frac{y^2}{81}\right) = \left(\frac{x^2}{4} + \frac{y^2}{9}\right)^2$$



Exercise 6.3

1. The foci are $(-1, 0)$ and $(7, 0)$.

The distance between the foci is $2ae = 8$ or $ae = 4$.

Since $e = 1/2$, we have $a = 8$.

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$\text{or } b^2 = 48$$

$$\text{or } b = 4\sqrt{3}$$

The center of the ellipse is the midpoint of the line joining two foci. Therefore, the coordinates of the center are $(3, 0)$.

Hence its equation is

$$\frac{(x-3)^2}{8^2} + \frac{(y-0)^2}{(4\sqrt{3})^2} = 1 \quad (1)$$

Thus, the parametric coordinates of a point on (1) are $(3 + 8 \cos \theta, 4\sqrt{3} \sin \theta)$.

2. The coordinates of any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angle is θ are $(a \cos \theta, b \sin \theta)$.

The coordinates of the endpoints of latus rectum are $(ae, \pm b^2/a)$. Therefore,

$$a \cos \theta = ae \text{ and } b \sin \theta = \pm \frac{b^2}{a}$$

$$\text{or } \tan \theta = \pm \frac{b}{ae}$$

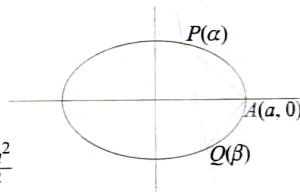
Hence, four points of latus rectum have eccentric angles $\tan^{-1}(b/ae)$, $\pi - \tan^{-1}(b/ae)$, $\pi + \tan^{-1}(b/ae)$, and $2\pi - \tan^{-1}(b/ae)$ in the first, second, third, and fourth quadrants, respectively.

3. $m_{AP} \times m_{QA} = -1$

$$\text{or } \frac{b \sin \alpha}{a \cos \alpha - a} \times \frac{b \sin \beta}{a \cos \beta - a} = -1$$

$$\text{or } \frac{4 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2}}{(2 \sin^2 \frac{\alpha}{2})(2 \sin^2 \frac{\beta}{2})} = -\frac{a^2}{b^2}$$

$$\text{or } \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = -\frac{b^2}{a^2}$$



4. The area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is $\pi ab = 4\pi$ (given). Therefore, $ab = 4$.

Consider point

$$A(a \cos \theta, b \sin \theta)$$

on the ellipse.

Now, the area of rectangle

$ABCD$ inscribed in the ellipse is

$$\Delta = 4(a \cos \theta)(b \sin \theta)$$

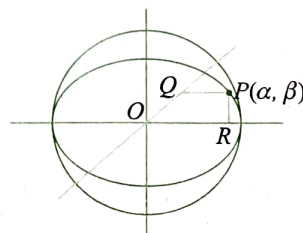
$$\text{or } \Delta = 2ab \sin 2\theta$$

$$\therefore \Delta_{\max} = 2ab = 8$$

5. The equation of the ellipse is

$$\frac{x^2}{3} + \frac{y^2}{4} = 1$$

Let point P be $(\sqrt{3} \cos \theta, 2 \sin \theta)$, $\theta \in (0, \pi/2)$.



Clearly, line PQ is $y = 2 \sin \theta$, line PR is $x = \sqrt{3} \cos \theta$, line OQ is $y = x$, and Q is $(2 \sin \theta, 2 \sin \theta)$.

Z = Area of region $PQORP$ (trapezium)

$$\begin{aligned} &= \frac{1}{2} (OR + PQ)PR \\ &= \frac{1}{2} (\sqrt{3} \cos \theta + (\sqrt{3} \cos \theta - 2 \sin \theta))2 \sin \theta \\ &= (2\sqrt{3} \cos \theta \sin \theta - 2 \sin^2 \theta) \\ &= (\sqrt{3} \sin 2\theta + \cos 2\theta - 1) \\ &= 2 \cos \left(2\theta - \frac{\pi}{3} \right) - 1 \end{aligned}$$

which is maximum when $\cos \left(\theta - \frac{\pi}{3} \right)$ is maximum

$$\therefore 2\theta - \frac{\pi}{3} = 0$$

$$\text{or } \theta = \frac{\pi}{6}$$

Hence, point P is $(3/2, 1)$.

6. Let $P \equiv (2\sqrt{2} \cos \theta, 2 \sin \theta)$

and $Q \equiv \left(2\sqrt{2} \cos \left(\frac{\pi}{2} + \theta \right), 2 \sin \left(\frac{\pi}{2} + \theta \right) \right)$

or $Q \equiv (-2\sqrt{2} \sin \theta, 2 \cos \theta)$

$$\begin{aligned} (PQ)^2 &= 8(\cos \theta + \sin \theta)^2 + 4(\sin \theta - \cos \theta)^2 \\ &= 12 + 4 \sin 2\theta \end{aligned}$$

$$\therefore (PQ)_{\max} = 4$$

7. We have points on the ellipse $P(3 \cos \theta, 2 \sin \theta)$ and $Q(-3 \sin \theta, 2 \cos \theta)$.

Let $M(h, k)$ be the midpoint of PQ .

$$\therefore h = \frac{3 \cos \theta - 3 \sin \theta}{2} \text{ and } k = \frac{2 \sin \theta + 2 \cos \theta}{2}$$

$$\therefore \cos \theta - \sin \theta = \frac{2h}{3} \text{ and } k = \sin \theta + \cos \theta$$

Squaring and adding, we get

$$\frac{4h^2}{9} + k^2 = 2$$

or $\frac{4x^2}{9} + y^2 = 2$, which is the required locus.

8. We have ellipse, $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$$a^2 e^2 = a^2 - b^2 = 4 - 3 = 1$$

Thus, foci are $S(1, 0)$ and $S'(-1, 0)$.

Variable point P on the ellipse is $(2 \cos \theta, \sqrt{3} \sin \theta)$.

Let centroid of triangle PSS' be $R(h, k)$.

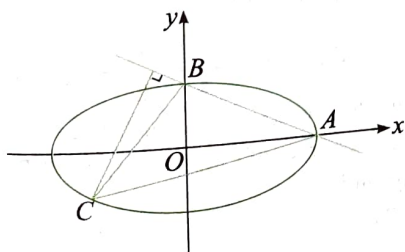
$$\therefore 3h = 1 - 1 + 2 \cos \theta \text{ and } 3k = 0 + 0 + \sqrt{3} \sin \theta$$

$$\therefore \cos \theta = \frac{3h}{2} \text{ and } \sin \theta = \sqrt{3} k$$

Squaring and adding, we get

$$\frac{9x^2}{4} + 3y^2 = 1, \text{ which is required locus.}$$

9.



Clearly, line $x + 2y = 1$ cuts the ellipse $\frac{x^2}{1} + \frac{y^2}{1/4} = 1$ at $A(1, 0)$ and $B(0, 1/2)$.

Let point C on the ellipse be $(\cos \theta, (\sin \theta)/2)$.

Now, area of triangle ABC is maximum, if distance of C from AB is maximum.

Distance of C from AB ,

$$d = \left| \frac{\cos \theta + \sin \theta - 1}{\sqrt{5}} \right|$$

Maximum value of d occurs when $\cos \theta = \sin \theta = -\frac{1}{\sqrt{2}}$.

$$\therefore C \equiv \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}} \right)$$

Exercise 6.4

1. We know that the line $y = mx + c$ is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{if } c^2 = a^2 m^2 + b^2.$$

Then comparing the given line $x \cos \alpha + y \sin \alpha = p$ with $y = mx + c$, we have $c = p/\sin \alpha$, $m = -\cos \alpha/\sin \alpha$.

So, the given line will be a tangent if

$$\frac{p^2}{\sin^2 \alpha} = a^2 \frac{\cos^2 \alpha}{\sin^2 \alpha} + b^2$$

$$\text{or } p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

2. For a given ellipse, the equation of tangent whose slope is m is y

$$= mx + \sqrt{18m^2 + 32}.$$

For $m = -4/3$, the tangent is

$$\begin{aligned} y &= -\frac{4}{3}x + \sqrt{18\left(\frac{16}{9}\right) + 32} \\ &= -\frac{4}{3}x + 8 \end{aligned}$$

$$\text{or } 4x + 3y = 24$$

This intersects the major and minor axes at $A(6, 0)$ and $B(0, 8)$, respectively.

Then, the area of $\triangle ACB$ is $(1/2)(6)(8) = 24$.

3. The equation of any tangent to the given ellipse is

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

If it touches $x^2 + y^2 = r^2$, then

$$\sqrt{a^2 m^2 + b^2} = r\sqrt{1 + m^2}$$

$$\text{or } m^2(a^2 - r^2) = r^2 - b^2$$

$$\text{or } m = \sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$$

4. The equation of tangent to the given ellipse at point $P(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

The intercepts of line on the axes are $a/\cos \theta$ and $b/\sin \theta$.

Given that

$$\frac{a}{\cos \theta} = \frac{b}{\sin \theta} = l \text{ or } \cos \theta = \frac{a}{l} \text{ and } \sin \theta = \frac{b}{l}$$

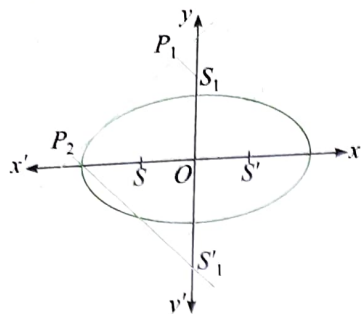
$$\text{or } \cos^2 \theta + \sin^2 \theta = \frac{a^2}{l^2} + \frac{b^2}{l^2}$$

$$\text{or } l^2 = a^2 + b^2$$

$$\text{or } l = \sqrt{a^2 + b^2}$$

5. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Then, the distance of a focus from the center is

$$ae = a \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{a^2 - b^2}$$

so that the two points on the minor axis are

$$S_1(0, \sqrt{a^2 - b^2}) \text{ and } S_1'(0, -\sqrt{a^2 - b^2})$$

Now, any tangent to the ellipse is

$$y = mx + \sqrt{a^2 m^2 + b^2}, \text{ where } m \text{ is a parameter}$$

The sum of the squares of the perpendiculars on this tangent from the two points S_1 and S_1' is

$$\begin{aligned} & \left(\frac{\sqrt{a^2 - b^2} - \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}} \right)^2 + \left(\frac{-\sqrt{a^2 - b^2} - \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}} \right)^2 \\ &= \frac{2(a^2 - b^2 + a^2 m^2 + b^2)}{1 + m^2} = 2a^2 \text{ (constant)} \end{aligned}$$

6. Let $P(a \cos \theta, b \sin \theta)$ be any point on the ellipse.

The equation of tangent at $P(\theta)$ is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \quad (1)$$

Tangent meets the major axis at $T(a \sec \theta, 0)$.

Applying sine rule in ΔPST , we get

$$\begin{aligned} \frac{PS}{\sin(\pi - \alpha)} &= \frac{ST}{\sin \beta} \\ \frac{a(1 - e \cos \theta)}{\sin \alpha} &= \frac{a(\sec \theta - e)}{\sin \beta} \end{aligned}$$

$$\text{or } \frac{1 - e \cos \theta}{\sin \alpha} = \frac{1 - e \cos \theta}{\cos \theta \sin \beta}$$

$$\text{or } \cos \theta = \frac{\sin \alpha}{\sin \beta}$$

The slope of tangent is

$$-\frac{b}{a} \cot \theta = \tan \alpha$$

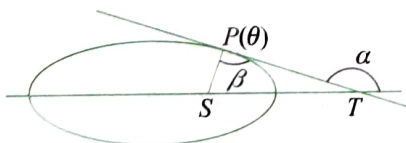
$$\text{or } \frac{b^2}{a^2} = \tan^2 \alpha (\sec^2 \theta - 1)$$

$$= \tan^2 \alpha \left(\frac{\sin^2 \beta - \sin^2 \alpha}{\sin^2 \alpha} \right)$$

$$= \frac{\sin^2 \beta - \sin^2 \alpha}{\cos^2 \alpha}$$

$$\text{or } e = \sqrt{1 - \frac{\sin^2 \beta - \sin^2 \alpha}{\cos^2 \alpha}}$$

$$= \sqrt{\frac{1 - \sin^2 \beta}{\cos^2 \alpha}} = \frac{\cos \beta}{\cos \alpha}$$



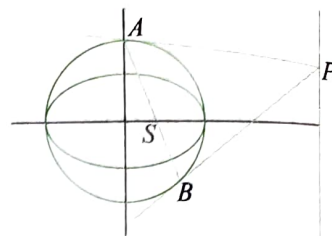
7. Let the point on the directrix be $(a/e, k)$. Then the equation of AB will be

$$x\left(\frac{a}{e}\right) + yk = a^2$$

Now, it will clearly pass through the focus of the ellipse as

$$(ae)\left(\frac{a}{e}\right) + k \cdot 0 = a^2$$

So, AB is a focal chord.



8. The equation of tangent to the ellipse at a given point is

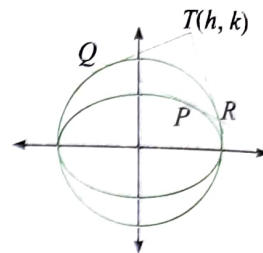
$$x\left(\frac{1}{\sqrt{2}}\right) + 2y\left(\frac{1}{2}\right) = 1$$

$$\text{or } x + \sqrt{2}y = \sqrt{2} \quad (1)$$

Now, QR is the chord of contact of the circle $x^2 + y^2 = 1$ with respect to the point $T(h, k)$. Then,

$$QR \equiv hx + ky = 1 \quad (2)$$

Equations (1) and (2) represent the same straight line. Then comparing the ratio of coefficients, we have $h/1 = k/\sqrt{2} = 1/\sqrt{2}$. Hence, $(h, k) \equiv (1/\sqrt{2}, 1)$.



9. Standard Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Equation of tangent to ellipse having slope m is

$$y = mx \pm \sqrt{a^2 m^2 + b^2} \quad (1)$$

Given equation of tangent is $2px + y \sqrt{5 - 6p^2} = 1$.

Comparing, we get

$$m = -\frac{2p}{\sqrt{5 - 6p^2}} \text{ and } a^2 m^2 + b^2 = \left(\frac{1}{\sqrt{5 - 6p^2}} \right)^2$$

$$\Rightarrow a^2 \frac{4p^2}{(5 - 6p^2)} + b^2 = \frac{1}{5 - 6p^2}$$

$$\Rightarrow 4a^2 p^2 + b^2 (5 - 6p^2) - 1 = 0$$

$$\Rightarrow p^2 (4a^2 - 6b^2) + 5b^2 - 1 = 0$$

Equation (1) should be true for all values of $p \in \left[-\sqrt{\frac{5}{6}}, \sqrt{\frac{5}{6}} \right]$.
 $\therefore 4a^2 = 6b^2$ and $5b^2 - 1 = 0$

$$\Rightarrow a^2 = \frac{3}{10} \text{ and } b^2 = \frac{1}{5}$$

$$\Rightarrow e = \sqrt{1 - \frac{10}{5 \times 3}} = \sqrt{\frac{5}{15}} = \frac{1}{\sqrt{3}}$$

10. When the quadrilateral is square, all the four vertices of square would lie on the director circle of the ellipse.

So, square is inscribed in the circle $x^2 + y^2 = a^2 + b^2$.

$\therefore$ Area of square = $2(a^2 + b^2) = 2(9 + 4) = 26$ sq. unit

11. Equation of a tangent to the circle $x^2 + y^2 = 1$ is

$$x \cos \theta + y \sin \theta = 1 \quad (1)$$

This line meets the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at P and Q .

Let the tangents to ellipse at P and Q intersect at $R(h, k)$.

PQ is chord of contact of ellipse w.r.t. point R .

So, its equation is

$$hx + 2ky = 4 \quad (2)$$

Equations (1) and (2) represent the same straight line.
 $\therefore h = 4 \cos \theta, k = 2 \sin \theta$

Eliminating θ , we get locus as $\frac{x^2}{16} + \frac{y^2}{4} = 1$, which is ellipse.

$$e = \sqrt{\frac{16-4}{16}} = \frac{\sqrt{3}}{2}$$

12. The equations of the chords of contact of tangents drawn from (x_1, y_1) and (x_2, y_2) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(1)

$$\text{and } \frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$$

(2)

It is given that (1) and (2) are at right angles. Therefore,

$$\frac{-b^2 x_1}{a^2 y_1} \times \frac{-b^2 x_2}{a^2 y_2} = -1$$

$$\text{or } \frac{x_1 x_2}{y_1 y_2} = -\frac{a^4}{b^4}$$

13. The equation of the chord of contact is

$$\frac{x}{4} + \frac{y}{3} = 1$$

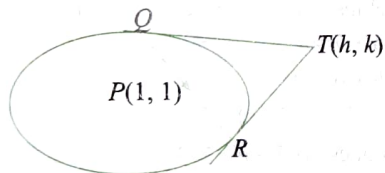
$$\therefore \text{Slope of line } EF = -\frac{3}{4}$$

The equation of line EF is

$$y = -\frac{3}{4}x + \sqrt{18}$$

Therefore, the distance of $(4, 3)$ from EF is $|24 - 4\sqrt{18}|/5$.

14.



Equation of chord having midpoint $P(1, 1)$ is

$$\frac{1 \cdot x}{16} + \frac{1 \cdot y}{9} - 1 = \frac{1}{16} + \frac{1}{9} - 1$$

$$\text{or } \frac{x}{16} + \frac{y}{9} = \frac{25}{144}$$

$$\text{or } 9x + 16y = 25$$

(1)

Also, line QR is chord of contact w.r.t. point $T(h, k)$.

Its equation is

$$\frac{hx}{16} + \frac{ky}{9} = 1$$

$$\text{or } 9hx + 16ky = 144$$

(2)

Equations (1) and (2) represent the same straight line.

$$\therefore \frac{9h}{9} = \frac{16k}{16} = \frac{144}{25}$$

$$\Rightarrow (h, k) = \left(\frac{144}{25}, \frac{144}{25} \right)$$

15. Centre of ellipse, $C \equiv \left(\frac{3+9}{2}, \frac{4+12}{2} \right) \equiv (6, 8)$

Foot of perpendicular from focus upon any tangent lies on the auxiliary circle.

Thus, $P(1, -4)$ lies on the auxiliary circle.

$$\text{So, } CP = 13 = a$$

Also, distance between foci,

$$SS' = \sqrt{(3-9)^2 + (4-12)^2} = 10 = 2ae$$

$$\therefore e = \frac{5}{13}$$

Exercise 6.5

1. Let the line $lx + my + n = 0$ be normal to the ellipse at the point $(a \cos \theta, b \sin \theta)$.

Now, the equation of the normal at $(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad (1)$$

Comparing this line with the line $lx + my + n = 0$, we have

$$\frac{a}{l \cos \theta} = \frac{-b}{m \sin \theta} = \frac{a^2 - b^2}{-n}$$

$$\text{or } \cos \theta = \frac{-an}{l(a^2 - b^2)}$$

$$\text{and } \sin \theta = \frac{bn}{m(a^2 - b^2)}$$

Squaring and adding, we get

$$\frac{a^2 n^2}{l^2 (a^2 - b^2)^2} + \frac{b^2 n^2}{m^2 (a^2 - b^2)^2} = 1$$

$$\text{or } \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

2. The equation of the normal at (x_1, y_1) to the given ellipse is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

Here, $x_1 = ae$ and $y_1 = b^2/a$.

So, the equation of the normal at the positive end of the latus rectum is

$$\frac{a^2 x}{ae} - \frac{b^2 y}{b^2/a} = a^2 - b^2 \quad [\because b^2 = a^2(1 - e^2)]$$

$$\text{or } \frac{ax}{e} - ay = a^2 e^2 \text{ or } x - ey - e^3 a = 0$$

3. Equation of tangent at point $P\left(\frac{\pi}{4}\right)$ is

$$\frac{x}{3} \cos \frac{\pi}{4} + \frac{y}{2} \sin \frac{\pi}{4} = 1$$

$$\text{or } 2x + 3y - 6\sqrt{2} = 0$$

Equation of normal at P is

$$3\sqrt{2}x - 2\sqrt{2}y - 5 = 0$$

Distance of centre from the tangent is $\frac{6\sqrt{2}}{\sqrt{13}}$.

Distance of centre from the normal is $\frac{5}{\sqrt{2}\sqrt{13}}$.

$$\therefore \text{Area of required rectangle} = \frac{6\sqrt{2}}{\sqrt{13}} \times \frac{5}{\sqrt{2}\sqrt{13}} = \frac{30}{13} \text{ sq. unit}$$

4. Given normal is

$$4ax + 3by = 24 \quad (1)$$

Let this line be normal to the ellipse at $P(a \cos \theta, b \sin \theta)$.

Equation of normal to ellipse at $P(a \cos \theta, b \sin \theta)$ is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2 \quad (2)$$

Equations (1) and (2) are identical, so

$$\frac{\sec \theta}{4} = \frac{-\operatorname{cosec} \theta}{3} = \frac{a^2 - b^2}{24}$$

$$\therefore \cos \theta = \frac{6}{a^2 - b^2} \text{ and } \sin \theta = \frac{-8}{a^2 - b^2}$$

Squaring and adding, we get

$$\frac{64}{(a^2 - b^2)^2} + \frac{36}{(a^2 - b^2)^2} = 1$$

$$\Rightarrow 100 = (a^2 - b^2)^2$$

$$\Rightarrow a^2 - b^2 = 10 \quad (\because a > b)$$

$$\Rightarrow a^2 e^2 = 10$$

Hence, foci are $(\pm\sqrt{10}, 0)$.

5. Normal at point P meets the major axis and minor axes at Q and R , respectively.

$$\therefore \frac{PQ}{PR} = \frac{b^2}{a^2}$$

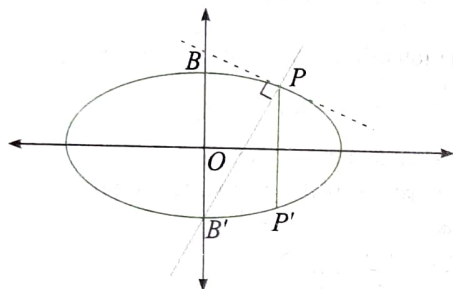
Given that $\frac{PQ}{PR} = \frac{2}{3}$

$$\therefore \frac{b^2}{a^2} = \frac{2}{3}$$

$$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow e = \frac{1}{\sqrt{3}}$$

6.



The equation of the normal at the extremity P of latus rectum is $x - ey = ae^3$.

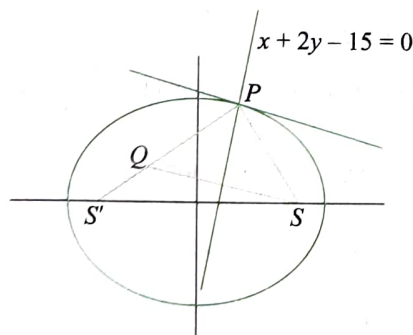
This passes through the extremity of the minor axis, i.e., $B'(0, -b)$. Then,

$$0 + eb - ae^3 = 0$$

$$\text{or } b = ae^2 \quad \text{or } b^2 = a^2 e^4$$

$$\text{or } a^2(1 - e^2) = a^2 e^4 \quad \text{or } e^4 + e^2 - 1 = 0$$

7.



We know that normal at P bisects $\angle SPS'$.

Thus, the image of S w.r.t. $x + 2y - 15 = 0$ lies on $S'P$.

Now, image of $S(3, 1)$ w.r.t. $x + 2y - 15 = 0$ is $Q(7, 9)$.

Equation of $S'Q$ is $x + y - 16 = 0$.

Clearly, from the figure, P is point of intersection of $S'Q$ and normal, which is $(17, -1)$.

8. Area of the circumscribing square is $2(a^2 + b^2)$ (using director circle concept).

$$\Rightarrow 2(a^2 + b^2) = 10$$

$$\Rightarrow a^2 + b^2 = 5$$

Also, maximum distance of a normal from centre is 1.

$$\therefore (a - b) = 1$$

From (1) and (2), we have

$$ab = 2 \text{ and } a + b = 3$$

Solving, we get $a = 2, b = 1$.

$$\therefore e = \sqrt{\frac{3}{4}}$$

Exercises

Single Correct Answer Type

1. (3) We have $PQ = BP$

$$\text{or } 2ae = \sqrt{a^2 e^2 + b^2} = \sqrt{a^2} = a$$

$$\text{or } e = \frac{1}{2}$$

2. (1) $2a(1 + e) = 15$

$$\text{or } 1 + e = \frac{3}{2}$$

$$\text{or } e = 0.5$$

3. (2) Here, $a^2 + 2 > a^2 + 1$

$$\therefore a^2 + 1 = (a^2 + 2)(1 - e^2)$$

$$\text{or } a^2 + 1 = (a^2 + 2) \frac{5}{6}$$

$$\text{or } 6a^2 + 6 = 5a^2 + 10$$

$$\text{or } a^2 = 10 - 6 = 4$$

$$\text{Latus rectum} = \frac{2(a^2 + 1)}{\sqrt{a^2 + 2}} = \frac{2 \times 5}{\sqrt{6}} = \frac{10}{\sqrt{6}}$$

4. (4) For the two ellipses to intersect at four distinct points, $a > 1$

$$\therefore b^2 - 5b + 7 > 1$$

$$\text{or } b^2 - 5b + 6 > 0$$

$$\text{or } b \in (-\infty, 2) \cup (3, \infty)$$

Therefore, b does not lie in $[2, 3]$.

5. (2) Clearly, locus is ellipse with eccentricity $e = \frac{1}{3}$.

Here, focus is $(2, 0)$ and directrix is $x - 18 = 0$.

Distance of focus from directrix = 16

$$\Rightarrow a/e - ae = 16$$

$$\therefore \frac{8}{3}a = 16$$

$$\therefore a = 6$$

$$\therefore b^2 = a^2(1 - e^2) = 36(1 - 1/9) = 32$$

$$\therefore L.R. = \frac{2b^2}{a} = \frac{32}{3}$$

6. (1) Vertices are $(-4, 1)$ and $(6, 1)$.

$$\therefore 2a = 10$$

Centre is $(1, 1)$.

Focus is at distance ' ae ' from the centre.

Thus, focus is $(1 + 5e, 1)$, which lies on $x - 2y = 2$.

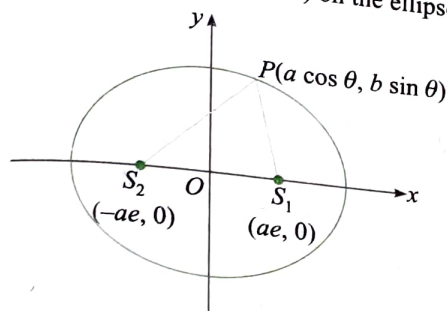
$$\therefore 1 + 5e - 2 = 2$$

$$\therefore e = 3/5$$

7. (1) We know that circumradius of triangle

$$R = \frac{\text{Product of all sides}}{4 \times \text{Area of triangle}}$$

Consider point $P(a \cos \theta, b \sin \theta)$ on the ellipse.

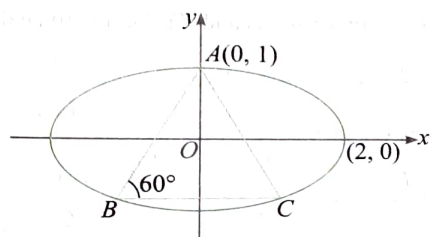


$$\begin{aligned} \therefore \text{Circumradius, } R &= \frac{a(1 - e \cos \theta) \times a(1 + e \cos \theta) \times 2ae}{4 \times \frac{1}{2} b \sin \theta \times 2ae} \\ &= \frac{a^2(1 - e^2 \cos^2 \theta)}{2b \sin \theta} \\ &= \frac{a^2(1 - e^2(1 - \sin^2 \theta))}{2b \sin \theta} \\ &= \frac{a^2}{2b} \left(e^2 \sin \theta + \frac{b^2}{a^2} \operatorname{cosec} \theta \right) \\ &\geq \frac{a^2}{2b} \times \frac{2be}{a} \end{aligned}$$

$$\therefore R_{\min} = \frac{a^2}{2b} \times \frac{2be}{a} = ae$$

8. (3) We have $\frac{x^2}{4} + y^2 = 1$

(1)



Equation of the side AB is

$$y - 1 = \sqrt{3}(x - 0)$$

$$\Rightarrow y = \sqrt{3}x + 1$$

(2)

Solving (1) and (2), we get

$$\frac{x^2}{4} + (\sqrt{3}x + 1)^2 = 1$$

$$\Rightarrow x^2 + 4(3x^2 + 1 + 2\sqrt{3}x) = 4$$

$$\Rightarrow 13x^2 + 8\sqrt{3}x = 0$$

$$\Rightarrow x = 0 \text{ or } x = -\frac{8\sqrt{3}}{13}$$

Putting $x = -\frac{8\sqrt{3}}{13}$ in (2), we get

$$y = -\sqrt{3} \left(\frac{8\sqrt{3}}{13} \right) + 1 = 1 - \frac{24}{13} = -\frac{11}{13}$$

$$\text{Hence, } B \equiv \left(-\frac{8\sqrt{3}}{13}, -\frac{11}{13} \right)$$

$$\therefore C \equiv \left(\frac{8\sqrt{3}}{13}, -\frac{11}{13} \right)$$

$$\therefore BC = \frac{16\sqrt{3}}{13}$$

9. (1) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle be $x^2 + y^2 = a^2e^2$.

Radius of circle = ae

One of the points of intersection of the circle and the ellipse is

$$\left(\frac{a}{e}\sqrt{2e^2 - 1}, \frac{a}{e}(1 - e^2) \right)$$

$$\text{Now, area of } \triangle PF_1F_2 = \frac{1}{2} \begin{vmatrix} \frac{a}{e}\sqrt{2e^2 - 1} & \frac{a}{e}(1 - e^2) & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix} = 30 \quad (\text{Given})$$

$$\text{or } \frac{1}{2} \left| \frac{a}{e}(1 - e^2)(2ae) \right| = 30$$

$$\text{or } a^2(1 - e^2) = 30$$

$$\text{or } a^2e^2 = a^2 - 30 = \left(\frac{17}{2} \right)^2 - 30 = \frac{169}{4}$$

$$\text{or } 2ae = 13$$

10. (2) Since there are exactly two points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose distance from the center is the same, the points would either be the endpoints of the major axis or of the minor axis.

But $\sqrt{(a^2 + 2b^2)/2} > b$. So, the points are the vertices of the major axis. Hence,

$$a = \sqrt{\frac{a^2 + 2b^2}{2}}$$

$$\text{or } a^2 = 2b^2$$

$$\text{or } e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{1}{\sqrt{2}}$$

11. (3) Let $p = 3h + 2$ and $q = k$

$$\text{or } h = \frac{p-2}{3} \text{ and } k = q$$

Since (h, k) lies on $x^2 + y^2 = 1$, we have

$$h^2 + k^2 = 1$$

$$\text{or } \left(\frac{p-2}{3} \right)^2 + q^2 = 1$$

The locus is

$$\left(\frac{x-2}{3} \right)^2 + y^2 = 1$$

which has eccentricity

$$e = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

12. (2) The radius of the circle having SS' as diameter is $r = ae$. If it cuts an ellipse, then

$$r > b$$

$$\text{or } ae > b$$

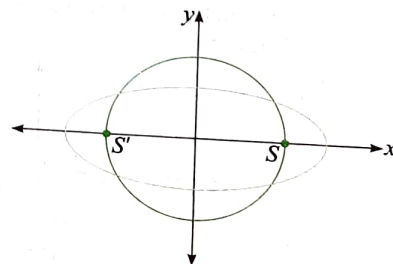
$$\text{or } e^2 > \frac{b^2}{a^2}$$

$$\text{or } e^2 > 1 - e^2$$

$$\text{or } e^2 > \frac{1}{2}$$

$$\text{or } e > \frac{1}{\sqrt{2}}$$

$$\text{or } e \in \left(\frac{1}{\sqrt{2}}, 1 \right)$$



13. (2) Let $Q \equiv \alpha + i\beta$.

Given that $|z| = 2$, where $z = x + iy$.

$$\therefore x^2 + y^2 = 4$$

$$\text{Now, } \alpha + i\beta = z + \frac{1}{z} = (x + iy) + \frac{1}{x + iy}$$

$$= (x + iy) + \left(\frac{x - iy}{4} \right) = \frac{5x}{4} + \frac{3iy}{4}$$

$$\therefore \alpha = \frac{5x}{4} \text{ and } \beta = \frac{3y}{4}$$

Since $x^2 + y^2 = 4$,

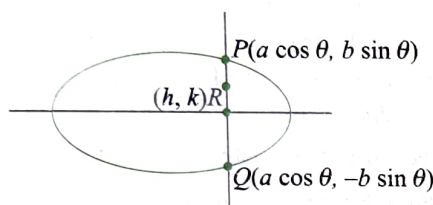
$$\frac{16\alpha^2}{25} + \frac{16\beta^2}{9} = 4$$

So, locus of point Q is $\frac{x^2}{25} + \frac{y^2}{9} = \frac{1}{4}$.

Eccentricity of this conic is given by

$$e^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

14. (1)



Let $P \equiv (a \cos \theta, b \sin \theta)$, $Q \equiv (a \cos \theta, -b \sin \theta)$.

$$PR : RQ = 1 : 2$$

$$\therefore h = a \cos \theta$$

$$\text{or } \cos \theta = \frac{h}{a} \quad (1)$$

$$\text{and } k = \frac{b}{3} \sin \theta$$

$$\text{or } \sin \theta = \frac{3k}{b} \quad (2)$$

On squaring and adding (1) and (2), we get

$$\frac{x^2}{a^2} + \frac{9y^2}{b^2} = 1$$

15. (1) Let (h, k) be the midpoint of the chord $7x + y - 1 = 0$. Then,

$$\frac{hx}{1} + \frac{ky}{7} = \frac{h^2}{1} + \frac{k^2}{7} \quad (1)$$

$$\text{and } 7x + y = 1 \quad (2)$$

represent the same straight line. Therefore,

$$\frac{h}{7} = \frac{k}{7} \text{ or } h = k$$

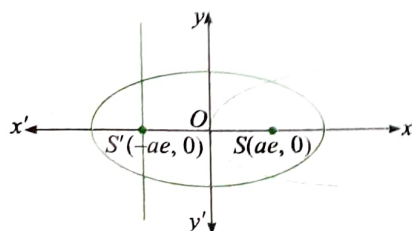
Thus, the equation of the line joining $(0, 0)$ and (h, k) is $y - x = 0$.

16. (3) The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The equation of the parabola with focus $S(ae, 0)$ and directrix $x + ae = 0$ is $y^2 = 4aex$.

Now, the length of latus rectum of the ellipse is $2b^2/a$ and that of the parabola is $4ae$.



For the two latus recta to be equal, we get

$$\frac{2b^2}{a} = 4ae$$

$$\text{or } \frac{2a^2(1 - e^2)}{a} = 4ae$$

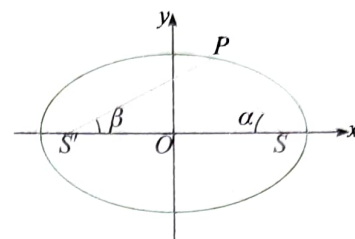
$$\text{or } 1 - e^2 = 2e$$

$$\text{or } e^2 + 2e - 1 = 0$$

$$\text{Therefore, } e = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$$

$$\text{Hence, } e = \sqrt{2} - 1$$

17. (3)



$$SP + S'P = 2a$$

In $\triangle SPS'$,

$$\text{Perimeter, } 2s = SP + S'P + SS' = 2a + 2ae = 2a(1 + e)$$

$$\text{Now, } \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-c)}} \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}$$

$$= \frac{s-c}{s} = \frac{2s-2c}{2s} = \frac{2a(1+e)-2c}{2a(1+e)}$$

$$= \frac{2a+2ae-4ae}{2a(1+e)} = \frac{2a(1-e)}{2a(1+e)}$$

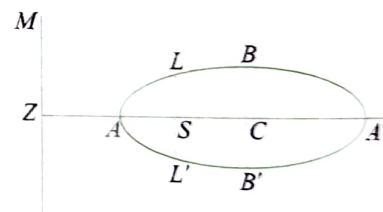
$$\therefore \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{1-e}{1+e}$$

18. (2) Let S be the given focus and ZM be the given directrix. Then,

$$SZ = \frac{a}{e} - ae$$

$$= \frac{a}{e} (1 - e^2)$$

$$= \frac{b^2}{ae} = k \text{ (say)}$$



Now, take SC as the x -axis and LSL' as the y -axis. Let (x, y) be the coordinates of B with respect to these axes. Then, $x = SC = ae$ and $y = CB = b$. Hence,

$$\frac{y^2}{x} = \frac{b^2}{ae} = SZ$$

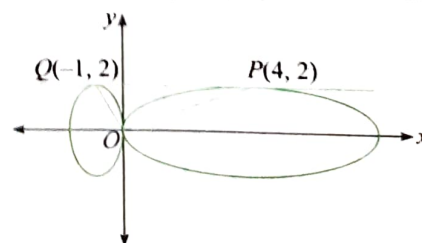
which is constant.

Therefore, $y^2 = kx$ is the required locus which is a parabola.

$$19. (4) \frac{(x-4)^2}{25} + \frac{y^2}{4} = 1 \text{ and } (x+1)^2 + \frac{y^2}{4} = 1$$

$$\text{Clearly, } m_{OP} \cdot m_{OQ} = -1$$

Therefore, OP and OQ are perpendicular to each other.



$$20. (2) (5x-10)^2 + (5y+15)^2 = \frac{(3x-4y+7)^2}{4}$$

$$\text{or } (x-2)^2 + (y+3)^2 = \left(\frac{1}{2} \frac{3x-4y+7}{5}\right)^2$$

$$\text{or } \sqrt{(x-2)^2 + (y+3)^2} = \frac{1}{2} \frac{|3x-4y+7|}{5}$$

It is an ellipse, whose focus is $(2, -3)$, directrix is $3x - 4y + 7 = 0$, and eccentricity is $1/2$.

Length of perpendicular from the focus to the directrix is

$$\frac{|3 \times 2 - 4(-3) + 7|}{5} = 5$$

$$\text{or } \frac{a}{e} - ae = 5$$

$$\text{or } 2a - \frac{a}{2} = 5$$

$$\text{or } a = \frac{10}{3}$$

So, the length of the major axis is $20/3$.

$$21. (1) \text{ Foci are } S(-1, -1) \text{ and } S'(0, -2).$$

Thus, centre is $C(-1/2, -3/2)$.

$$2ae = SS' = \sqrt{2}$$

$$\therefore a = \sqrt{2}$$

Now, directrix is at distance $\frac{a}{e}$ from the centre.

So, we have to find equation of line at distance $2\sqrt{2}$ from $C(-1/2, -3/2)$.

Also, slope of axis is equal to -1 .

Thus, equation of directrix is $y = x + c$.

Its distance from centre is $2\sqrt{2}$.

$$\therefore \frac{\left| -\frac{1}{2} + \frac{3}{2} + c \right|}{\sqrt{2}} = 2\sqrt{2}$$

$$\therefore c = -5 \text{ or } 3$$

Therefore, directrix is $x - y + 3 = 0$ (as it is nearer to focus S).

$$22. (2) \text{ The equation of conic through the point of intersection of given two ellipses is}$$

$$\left(\frac{x^2}{4} + y^2 - 1\right) + \lambda \left(\frac{x^2}{a^2} + y^2 - 1\right) = 0$$

$$\text{or } x^2 \left(\frac{1}{4} + \frac{\lambda}{a^2}\right) + y^2(1 + \lambda) = 1 + \lambda$$

$$\text{or } x^2 \left(\frac{a^2 + 4\lambda}{4a^2(1 + \lambda)}\right) + y^2 = 1$$

This equation is a circle if

$$\frac{a^2 + 4\lambda}{4a^2(1 + \lambda)} = 1$$

Therefore, the circle is $x^2 + y^2 = 1$.

$$23. (3) \text{ Here, the center of the ellipse is } (0, 0).$$

Let $P(r \cos \theta, r \sin \theta)$ be any point on the given ellipse. Then

$$r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + 2r^2 \sin \theta \cos \theta = 1$$

$$\text{or } r^2 = \frac{1}{\cos^2 \theta + 2 \sin^2 \theta + \sin 2\theta}$$

$$= \frac{1}{\sin^2 \theta + 1 + \sin 2\theta}$$

$$= \frac{2}{1 - \cos 2\theta + 2 + 2 \sin 2\theta}$$

$$= \frac{2}{3 - \cos 2\theta + 2 \sin 2\theta}$$

$$\text{or } r_{\max} = \frac{\sqrt{2}}{\sqrt{3} - \sqrt{5}}$$

$$24. (3) \text{ Let } P(5 \cos \theta, 4 \sin \theta) \text{ be any point on the ellipse. Then,}$$

$$SP = 5 + 5e \cos \theta$$

$$S'P = 5 - 5e \cos \theta$$

$$SP \cdot S'P = 25 - 25e^2 \cos^2 \theta$$

$$= 25 - 9 \cos^2 \theta = f(\theta) \text{ (say)}$$

$$\text{or } 16 \leq f(\theta) \leq 25$$

$$25. (1) \text{ Let } P(x, y) \text{ be the position of the man at any time.}$$

$$\text{Let } S(4, 0) \text{ and } S'(-4, 0) \text{ be the fixed flag post, with } C \text{ as the origin.}$$

Since $SP + S'P = 10$ m, i.e., a constant, the locus of P is an ellipse with S and S' as foci. Therefore,

$$ae = 4$$

$$\text{Also, } 2a = 10$$

$$\therefore e = \frac{4}{5}$$

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$\text{or } b^2 = 25 \left(1 - \frac{16}{25}\right) = 9$$

$$\text{or } b = 3$$

Hence, the area of the ellipse is $\pi ab = \pi \times 5 \times 3 = 15\pi$.

$$26. (1) \text{ Any point on the ellipse is } (2 \cos \theta, \sqrt{3} \sin \theta).$$

The focus on the positive x -axis is $(1, 0)$.

Given that

$$(2 \cos \theta - 1)^2 + 3 \sin^2 \theta = \frac{25}{16}$$

$$\text{or } \cos \theta = \frac{3}{4}$$

$$27. (1) \text{ Consider } P(\theta), Q\left(\theta + \frac{2\pi}{3}\right), \text{ and } R\left(\theta + \frac{4\pi}{3}\right). \text{ Then,}$$

$$P' \equiv (a \cos \theta, b \sin \theta)$$

$$Q' \equiv \left(a \cos\left(\theta + \frac{2\pi}{3}\right), b \sin\left(\theta + \frac{2\pi}{3}\right)\right)$$

$$\text{and } R' \equiv \left(a \cos\left(\theta + \frac{4\pi}{3}\right), b \sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Let the centroid of $\Delta P'Q'R'$ be (x', y') .

$$x' = a \left[\frac{\cos \theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)}{3} \right]$$

$$= \frac{a}{3} \left[\cos \theta + 2 \cos(\theta + \pi) \cos \frac{\pi}{3} \right] = 0$$

$$y' = \frac{a}{3} \left[\sin \theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) \right]$$

$$= \frac{a}{3} \left[\sin \theta + 2 \sin(\theta + \pi) \cos \frac{\pi}{3} \right] = 0$$

$$28. (1) \text{ Tangent at } P(a \cos \alpha, b \sin \alpha) \text{ is}$$

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$$

The distance of focus $S(ae, 0)$ from this tangent is

$$d_1 = \frac{|e \cos \alpha - 1|}{\sqrt{\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}}}$$

$$= \frac{1 - e \cos \alpha}{\sqrt{\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}}}$$

The distance of focus $S'(-ae, 0)$ from this line is

$$d_2 = \frac{1 + e \cos \alpha}{\sqrt{\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}}}$$

(1)

$$\text{or } \frac{d_1}{d_2} = \frac{1 - e \cos \alpha}{1 + e \cos \alpha}$$

$$\text{Now, } SP = a - ae \cos \alpha \text{ and } S'P = a + ae \cos \alpha$$

$$\text{or } \frac{SP}{S'P} = \frac{1 - e \cos \alpha}{1 + e \cos \alpha} = \frac{d_1}{d_2}$$

29. (2) Let m be the slope of the common tangent. Then,

$$\pm \sqrt{3} \sqrt{1+m^2} = \pm \sqrt{4m^2+1}$$

$$\text{or } 3 + 3m^2 = 4m^2 + 1$$

$$\text{or } m^2 = 2$$

$$\text{or } m = \pm \sqrt{2}$$

30. (4) Using condition of tangency, we get

$$c^2 = 4m^2 + 2 \geq 2 \quad \forall m \in R$$

$$\therefore c^2 \geq 2$$

$$\Rightarrow |c| \geq \sqrt{2}$$

31. (1) The equation of tangent is

$$\frac{x}{a} \frac{\sqrt{3}}{2} + \frac{y}{b} \frac{1}{2} = 1 \quad (1)$$

and the equation of tangent at the point $(a \cos \phi, b \sin \phi)$ is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1 \quad (2)$$

Comparing (1) and (2), we have

$$\cos \phi = \frac{\sqrt{3}}{2} \text{ and } \sin \phi = \frac{1}{2}$$

Hence, $\phi = \pi/6$.

32. (2) Given ellipse is $\frac{x^2}{9} + \frac{y^2}{27} = 1$.

Tangent to ellipse having slope m is $y = mx \pm \sqrt{a^2 m^2 + b^2}$.

Since tangent makes equal intercepts on the coordinate axes, $m = -1$.

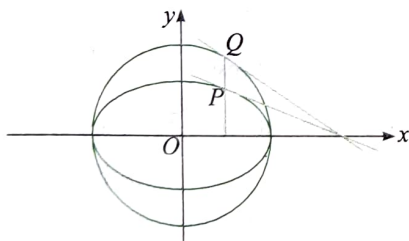
Therefore, equations of tangents are

$$y = -x \pm 6$$

$$\text{or } x + y \pm 6 = 0$$

$$\therefore \text{Length of perpendicular from } (0, 0) = \left| \frac{6}{\sqrt{2}} \right| = 3\sqrt{2}$$

33. (3)



Tangent to the ellipse at point $P(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \quad (1)$$

Tangent to the circle at point $Q(a \cos \theta, a \sin \theta)$ is

$$x \cos \theta + y \sin \theta = a \quad (2)$$

Equations (1) and (2) intersect at $(a/\cos \theta, 0)$ which lies on $y = 0$.

34. (4) The given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Then, $a^2 = 9$, $b^2 = 5$. Therefore,

$$e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

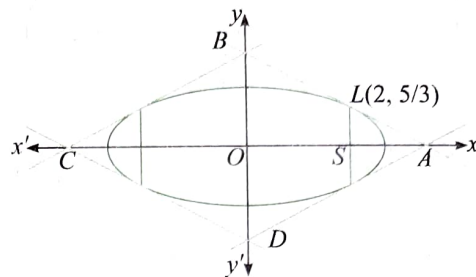
Hence, the endpoint of latus rectum in first quadrant is $L(2, 5/3)$.

The equation of tangent at L is

$$\frac{2x}{9} + \frac{y}{3} = 1$$

the tangent meets the x -axis at $A(9/2, 0)$ and the y -axis at $B(0, 3)$. Therefore,

$$\text{Area of } \triangle OAB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$$



By symmetry,

Area of quadrilateral = $4 \times (\text{Area of } \triangle OAB)$

$$= 4 \times \frac{27}{4} = 27 \text{ sq. units}$$

35. (1) Any tangent to ellipse

$$\frac{x^2}{2} + \frac{y^2}{1} = 1 \text{ is given by}$$

$$\frac{x \cos \theta}{\sqrt{2}} + y \sin \theta = 1$$

Let it meet axes at A and B .

$$\therefore A \equiv (\sqrt{2} \sec \theta, 0) \text{ and } B \equiv (0, \csc \theta).$$

Let $p(h, k)$ be the mid point of AB .

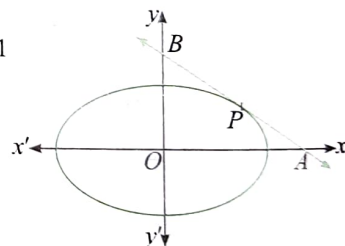
$$\text{Hence, } 2h = \sqrt{2} \sec \theta \text{ and } 2k = \csc \theta$$

Eliminating ' θ ', which is locus of P .

$$\left(\frac{1}{\sqrt{2}h} \right)^2 + \left(\frac{1}{2k} \right)^2 = 1$$

$$\text{or } \frac{1}{2h^2} + \frac{1}{4k^2} = 1$$

$$\text{or } \frac{1}{2x^2} + \frac{1}{4y^2} = 1$$



36. (1) Tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at $P(a \cos \theta, b \sin \theta)$ is given by

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

It meets the coordinate axes at $A(a \sec \theta, 0)$ and $B(0, b \csc \theta)$. Therefore,

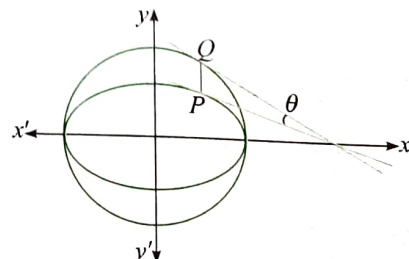
$$\text{Area of } \triangle OAB = \frac{1}{2} \times a \sec \theta \times b \csc \theta$$

$$\text{or } \Delta = \frac{ab}{\sin 2\theta}$$

For area to be minimum, $\sin 2\theta$ should be maximum and we know that the maximum value of $\sin 2\theta$ is 1. Therefore,

$$\Delta_{\max} = ab$$

37. (1)



Tangent to the ellipse at $P(a \cos \alpha, b \sin \alpha)$ is

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$$

Tangent to the circle at $Q(a \cos \alpha, a \sin \alpha)$ is

$$\cos \alpha x + \sin \alpha y = a$$

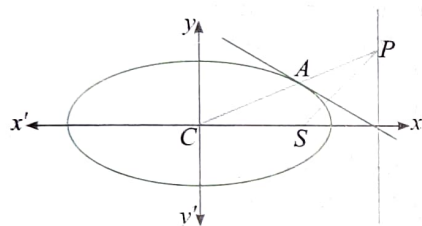
Now, the angle between the tangents is θ . Then,

$$\begin{aligned} \tan \theta &= \left| \frac{-\frac{b}{a} \cot \alpha - (-\cot \alpha)}{1 + \left(-\frac{b}{a} \cot \alpha\right)(-\cot \alpha)} \right| \\ &= \left| \frac{\cot \alpha \left(1 - \frac{b}{a}\right)}{1 + \frac{b}{a} \cot^2 \alpha} \right| = \left| \frac{a - b}{a \tan \alpha + b \cot \alpha} \right| \\ &= \left| \frac{a - b}{(\sqrt{a} \tan \alpha - \sqrt{b} \cot \alpha)^2 + 2\sqrt{ab}} \right| \end{aligned}$$

Now, the greatest value of the above expression is $\left| \frac{a - b}{2\sqrt{ab}} \right|$ when $\sqrt{a} \tan \alpha = \sqrt{b} \cot \alpha$. Therefore,

$$\theta_{\text{maximum}} = \tan^{-1} \left(\frac{a - b}{2\sqrt{ab}} \right)$$

38. (2)



Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and let $A \equiv (a \cos \theta, b \sin \theta)$.

The equation of AC will be

$$y = \frac{b}{a} \tan \theta x$$

Solving with $x = a/e$, we get

$$P \equiv \left(\frac{a}{e}, \frac{b}{e} \tan \theta \right)$$

Slope of tangent at A = $-\frac{b}{a} \tan \theta$

$$\text{Slope of PS} = \frac{\frac{b}{e} \tan \theta}{\frac{a}{e} - ae} = \frac{b \tan \theta}{a(1 - e^2)} = \frac{a}{b} \tan \theta$$

$$\text{So, } \alpha = \frac{\pi}{2}$$

39. (1) A tangent of slope 2 to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = 4x \pm \sqrt{4a^2 + b^2}$$

This is normal to the circle $x^2 + y^2 + 4x + 1 = 0$.

Therefore, (1) passes through $(-2, 0)$, i.e.,

$$\text{or } 0 = -4 \pm \sqrt{4a^2 + b^2}$$

$$\text{or } 4a^2 + b^2 = 16$$

Using AM $\geq$ GM, we get

$$\frac{4a^2 + b^2}{2} \geq \sqrt{4a^2 b^2}$$

$$\text{or } ab \leq 4$$

40. (4) Tangent to the ellipse having slope m is

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

If it passes through the point $P(h, k)$, then

$$k = mh + \sqrt{a^2 m^2 + b^2}$$

$$\text{or } (a^2 - h^2)m^2 + 2hkm + b^2 - k^2 = 0$$

Now, given $\tan \alpha + \tan \beta = \lambda$. Then,

$$m_1 + m_2 = \lambda$$

$$\text{or } \frac{-2hk}{a^2 - h^2} = \lambda$$

Therefore, the locus is $\lambda(x^2 - a^2) = 2xy$.

41. (3) Given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($b > a$) and a parabola $y^2 = 4ax$.

Differentiating ellipse w.r.t. x , we get

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

$$\therefore y' = -\frac{b^2 x}{a^2 y}$$

Differentiating parabola w.r.t. x , we get

$$\therefore y' = \frac{2a}{y}$$

Since curves intersect orthogonally, we have

$$-\frac{b^2 x}{a^2 y} \cdot \frac{2a}{y} = -1$$

$$\Rightarrow \frac{2b^2 x}{ay^2} = 1$$

$$\Rightarrow \frac{2b^2 x}{a(4ax)} = 1$$

$$\Rightarrow \frac{a^2}{b^2} = \frac{1}{2}$$

$$\Rightarrow e^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow e = \frac{1}{\sqrt{2}}$$

42. (3) Tangents to the ellipse at P and Q are, respectively,

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$$

(1)

$$\text{and } \frac{x}{a} \cos \beta + \frac{y}{b} \sin \beta = 1$$

(2)

Solving (1) and (2), we get

$$\begin{vmatrix} x & -y & 1 \\ \frac{\sin \alpha}{b} & 1 & \\ \frac{\sin \beta}{b} & 1 & \end{vmatrix} = \begin{vmatrix} \cos \alpha & 1 \\ \cos \beta & 1 \end{vmatrix} = \begin{vmatrix} \cos \alpha & \sin \alpha \\ \cos \beta & \sin \beta \end{vmatrix}$$

$$\text{or } x = \frac{a(\sin \alpha - \sin \beta)}{\sin(\beta - \alpha)}$$

$$\text{and } y = \frac{-b(\cos \alpha - \cos \beta)}{\sin(\beta - \alpha)}$$

$$\text{or } \frac{x \sin(\beta - \alpha)}{a} = \sin \alpha - \sin \beta$$

$$\text{and } \frac{y \sin(\beta - \alpha)}{b} = -(\cos \alpha - \cos \beta)$$

Squaring and adding, we get

$$\sin^2(\beta - \alpha) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = 2$$

or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2}{\sin^2 c}$ [where $\beta - \alpha = c$ (constant) given]
which is an ellipse.

43. (4) As in the above question, the point of intersection is

$$(h, k) \equiv \left(-\frac{a \cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, -\frac{b \sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \right)$$

It is given that $\alpha + \beta = c = \text{constant}$. Therefore,

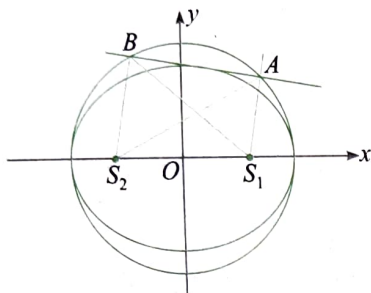
$$h = -\frac{a \cos \frac{c}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)} \text{ and } k = -\frac{b \sin \frac{c}{2}}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\text{or } \frac{h}{k} = \frac{a}{b} \cot\left(\frac{c}{2}\right)$$

$$\text{or } k = \frac{b}{a} \tan\left(\frac{c}{2}\right) h$$

Therefore, (h, k) lies on the straight line.

44. (1)



Let AB be the length of the tangent intercepted by auxiliary circle and S_1, S_2 be the foci.

Given that

$$\angle AS_1B = \angle AS_2B$$

$$\Rightarrow \angle S_1AB = \angle S_2AB = \frac{\pi}{2}$$

So, AS_1S_2B is a rectangle.

$$\therefore AB = S_1S_2 = 2ae = 2 \times 5 \times \frac{3}{5} = 6$$

45. (3) The equation of the tangent to the ellipse at $P(5 \cos \theta, 4 \sin \theta)$ is

$$\frac{x \cos \theta}{5} + \frac{y \sin \theta}{4} = 1$$

It meets the line $x = 0$ at $Q(0, 4 \operatorname{cosec} \theta)$.

The image of Q on the line $y = x$ is $R(4 \operatorname{cosec} \theta, 0)$.

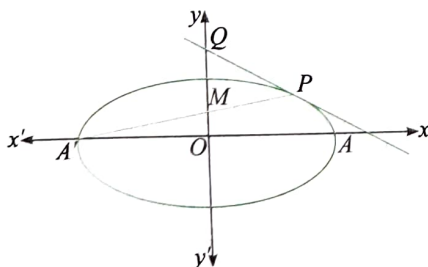
Therefore, the equation of the circle is

$$x(x - 4 \operatorname{cosec} \theta) + y(y - \operatorname{cosec} \theta) = 0$$

$$\text{i.e., } x^2 + y^2 - 4(x + y) \operatorname{cosec} \theta = 0$$

Therefore, each member of the family passes through the intersection of $x^2 + y^2 = 0$ and $x + y = 0$, i.e., the point $(0, 0)$.

46. (3)



Let point P be $(a \cos \theta, b \sin \theta)$.

The equation of the tangent at point P is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

Then point Q is $(0, b \operatorname{cosec} \theta)$.

The equation of chord $A'P$ is

$$y - 0 = \frac{b \sin \theta}{a \cos \theta + a} (x + a)$$

Putting $x = 0$, we have

$$y = \frac{b \sin \theta}{\cos \theta + 1}$$

Then,

$$\begin{aligned} OQ^2 - MQ^2 &= b^2 \operatorname{cosec}^2 \theta - \left(b \operatorname{cosec} \theta - \frac{b \sin \theta}{\cos \theta + 1} \right)^2 \\ &= \frac{2b^2}{\cos \theta + 1} - \frac{b^2 \sin^2 \theta}{(\cos \theta + 1)^2} \\ &= \frac{b^2}{\cos \theta + 1} \left(\frac{2 \cos \theta + 2 - \sin^2 \theta}{\cos \theta + 1} \right) \\ &= \frac{b^2}{\cos \theta + 1} \left(\frac{2 \cos \theta + 1 + \cos^2 \theta}{\cos \theta + 1} \right) = b^2 = 4 \end{aligned}$$

47. (4) Since sides of square are tangent to ellipse and perpendicular to each other, vertices lie on the director circle.

Equation of director circle is

$$x^2 + y^2 = (a^2 - 7) + (13 - 5a)$$

$$\text{or } x^2 + y^2 = a^2 - 5a + 6$$

Also, side of square is $a\sqrt{2}$.

Thus, radius of the circle is a .

$$\therefore a^2 - 5a + 6 = a^2$$

$$\text{or } a = 6/5$$

But for an ellipse, we must have $a^2 - 7 > 0$ and $13 - 5a > 0$.

$$\Rightarrow a \in (-\infty, -\sqrt{7}) \cup (\sqrt{7}, 0) \text{ and } a < \frac{13}{5}$$

$$\therefore a \neq \frac{6}{5}$$

Hence, no such a exists.

48. (4) Since the locus of the point of intersection of the tangent at the endpoints of a focal chord is directrix, the required locus is $x = \pm a/e$, which is a pair of straight lines.

49. (4) The equation of QR is $T = 0$ (chord of contact)

$$\frac{8x}{4} + \frac{27y}{9} = 1$$

$$\text{or } 2x + 3y = 1$$

(1)

Now, the equation of the pair of lines passing through the origin and points Q and R is given by

$$\left(\frac{x^2}{4} + \frac{y^2}{9} \right) = (2x + 3y)^2$$

[Making the equation of ellipse homogeneous using (1)]

$$\text{or } 9x^2 + 4y^2 = 36(4x^2 + 12xy + 9y^2)$$

$$\text{or } 135x^2 + 432xy + 320y^2 = 0$$

Therefore, the required angle is

$$\tan^{-1} \frac{2\sqrt{216^2 - 135 \times 320}}{455} = \tan^{-1} \frac{8\sqrt{2916 - 2700}}{455}$$

$$= \tan^{-1} \frac{8\sqrt{216}}{455} = \tan^{-1} \frac{48\sqrt{6}}{455}$$

50. (3) A cyclic parallelogram will be a rectangle or a square.

So, $\angle QPR = 90^\circ$.

Therefore, P lies on the director circle of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$.

Hence, $x^2 + y^2 = 25$ is the director circle of $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$. Then,

$$16 + b^2 = 25$$

$$\text{or } b^2 = 9$$

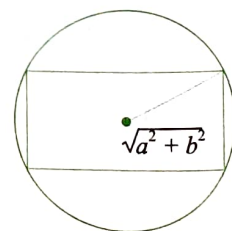
$$\text{Now, } a^2(1 - e^2) = 9$$

$$\text{or } 1 - e^2 = \frac{9}{16}$$

$$\text{or } e^2 = \frac{7}{16}$$

$$\text{or } e = \frac{\sqrt{7}}{4}$$

51. (4) Since mutually perpendicular tangents can be drawn from the vertices of the rectangle, all the vertices of the rectangle should lie on the director circle $x^2 + y^2 = a^2 + b^2$.



Let breadth = $2l$ and length = $4l$. Then,

$$l^2 + (2l)^2 = a^2 + b^2$$

$$\text{or } l^2 = \frac{a^2 + b^2}{5}$$

$$\text{or Area} = 4l \times 2l = 8 \left(\frac{a^2 + b^2}{5} \right)$$

52. (4) Let the centre of the ellipse be $C(x, y)$.

Clearly, axes are tangents to the ellipse which are perpendicular and intersect at origin O .

So, origin lies on the director circle.

Thus, OC is radius of the director circle.

$$\text{So, } OC^2 = a^2 + b^2 = 22 + 12$$

$$\text{or } x^2 + y^2 = 5$$

53. (4) For the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

the equation of director circle is $x^2 + y^2 = 25$. The director circle will cut the ellipse

$$\frac{x^2}{50} + \frac{y^2}{20} = 1$$

at four points. Hence,

Number of points = 4

$$54. (3) \sum_{i=1}^{10} (SP_i)(S'P_i) = 2560$$

$$\text{or } 10b^2 = 2560$$

$$\text{or } b^2 = 256$$

$$\text{or } b = 16$$

$$\text{Now, } 256 = 400(1 - e^2)$$

$$\text{or } 1 - e^2 = \frac{16}{25}$$

$$\text{or } e = \frac{3}{5}$$

55. (3) Normal at point $P(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2 \quad (1)$$

It meets the axes at

$$Q\left(\frac{(a^2 - b^2) \cos \theta}{a}, 0\right)$$

$$\text{and } R\left(0, -\frac{(a^2 - b^2) \sin \theta}{b}\right)$$

Let $T(h, k)$ be a midpoint of QR . Then,

$$2h = \frac{(a^2 - b^2) \cos \theta}{a}$$

$$\text{and } 2k = -\frac{(a^2 - b^2) \sin \theta}{b}$$

$$\text{or } \cos^2 \theta + \sin^2 \theta = \frac{4h^2 a^2}{(a^2 - b^2)^2} + \frac{4k^2 b^2}{(a^2 - b^2)^2} = 1$$

Therefore, the locus is

$$\frac{x^2}{\frac{(a^2 - b^2)^2}{4a^2}} + \frac{y^2}{\frac{(a^2 - b^2)^2}{4b^2}} = 1 \quad (2)$$

which is an ellipse, having eccentricity e' , given by

$$e'^2 = 1 - \frac{(a^2 - b^2)^2 / 4a^2}{(a^2 - b^2)^2 / 4b^2} = 1 - \frac{b^2}{a^2} = e^2$$

or $e' = e$

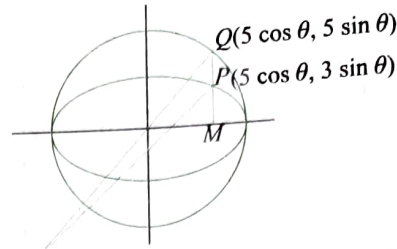
Note: In (2), $\frac{(a^2 - b^2)}{4a^2} < \frac{(a^2 - b^2)}{4b^2}$. Hence, the x -axis is the minor axis.

56. (3) The equation of normal to the ellipse at P is (1)

$$5x \sec \theta - 3y \csc \theta = 16$$

The equation of normal to the circle $x^2 + y^2 = 25$ at point Q is (2)

$$y = x \tan \theta$$



The eliminating θ from (1) and (2), we get $x^2 + y^2 = 64$.

$$57. (3) \frac{x^2}{169} + \frac{y^2}{25} = 1$$

The equation of normal at the point $(13 \cos \theta, 5 \sin \theta)$ is

$$\frac{13x}{\cos \theta} - \frac{5y}{\sin \theta} = 144$$

It passes through $(0, 6)$. Therefore,

$$0 - 30 = 144 \sin \theta$$

$$\text{or } \sin \theta = -\frac{5}{24}$$

$$\text{or } \theta = 2\pi - \sin^{-1}\left(\frac{5}{24}\right) \text{ and } \pi + \sin^{-1}\frac{5}{24}$$

Also, the y -axis is one of the normals.

58. (3) The equation of the normal to the given ellipse at the point $P(a \cos \theta, b \sin \theta)$ is $ax \sec \theta - by \csc \theta = a^2 - b^2$. Then,

$$y = \left(\frac{a}{b} \tan \theta\right)x - \frac{(a^2 - b^2)}{b} \sin \theta \quad (1)$$

$$\text{Let } \frac{a}{b} \tan \theta = m$$

so that

$$\sin \theta = \frac{bm}{\sqrt{a^2 + b^2 m^2}}$$

Hence, the equation of the normal equation (1) becomes

$$y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2 m^2}}$$

$$\therefore m \in R, \text{ as } m = \frac{a}{b} \tan \theta \in R$$

Multiple Correct Answers Type

1. (1), (3)

$$r^2 - r - 6 > 0 \text{ and } r^2 - 6r + 5 > 0$$

$$\text{or } (r - 3)(r + 2) > 0 \text{ and } (r - 1)(r - 5) > 0$$

$$\text{or } (r < -2 \text{ or } r > 3) \text{ and } (r < 1 \text{ or } r > 5)$$

$$\text{i.e., } r < -2 \text{ or } r > 5$$

$$\text{Also, } r^2 - r - 6 \neq r^2 - 6r + 5$$

$$\text{or } r \neq \frac{11}{5}$$

2. (1), (2), (3), (4)

$$(\sqrt{3}x - 3\sqrt{3})^2 + (2y + 4)^2 = k$$

So, no locus for $k < 0$, ellipse for $k > 0$, and point for $k = 0$.

3. (1), (2), (3)

$$3x^2 + 2y^2 + 6x - 8y + 5 = 0$$

$$\text{or } \frac{(x + 1)^2}{2} + \frac{(y - 2)^2}{3} = 1$$

Therefore, the center is $(-1, 2)$ and the ellipse is vertical (since $b > a$).

$$a^2 = 2, b^2 = 3$$

$$\text{Now, } 2 = 3(1 - e^2)$$

or $e = \frac{1}{\sqrt{3}}$

The foci are $(-1, 2 \pm be)$ and $(-1, 2 \pm 1)$, i.e., $(-1, 3)$ and $(-1, 1)$. The directrices are $y = 2 \pm b/e$ or $y = 5$ and $y = -1$.

4. (2), (3)

The given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$b^2 = a^2(1 - e^2)$$

or $e = \frac{\sqrt{3}}{2}$

Hence, the endpoints P and Q of the latus rectum are given by

$$P \equiv \left(\sqrt{3}, -\frac{1}{2}\right)$$

and $Q \equiv \left(-\sqrt{3}, -\frac{1}{2}\right)$

(Given $y_1, y_2 < 0$)

The midpoint of PQ is

$$R \equiv \left(0, -\frac{1}{2}\right)$$

length of latus rectum, $PQ = 2\sqrt{3}$

Hence, two parabolas are possible whose vertices are

$$S\left(0, +\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \text{ and } T\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$$

the equations of the parabolas are

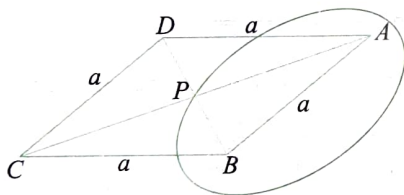
$$x^2 = 2\sqrt{3}\left(y + \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

and $x^2 = -2\sqrt{3}\left(y - \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

or $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

5. (1), (2), (4)



$$AB = BC = CD = DA = a$$

Since $\triangle PAB, \triangle PBC, \triangle PCD, \triangle PDA$ are congruent,

$$e_1 = e_2 = e_3 = e_4 = \frac{a}{PA + PB}$$

6. (1), (3)

Foot of perpendicular from one end of latus rectum $L(3, 4)$ on major axis which is focus is given by

$$\frac{x-3}{1} = \frac{y-4}{-2} = \frac{-(3-2(4)-5)}{1^2 + (-2)^2} = 2$$

$$\therefore S(x, y) \equiv S(5, 0)$$

Another focus is image of S in the line $2x + y + 10 = 0$.

$$\frac{x-5}{2} = \frac{y-0}{1} = \frac{-2(2(5)+0+10)}{2^2 + 1^2} = -8$$

$$\therefore S'(x, y) \equiv S'(-11, -8)$$

7. (3), (4)

The equation of the line joining θ and ϕ is

$$\frac{x}{5} \cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{3} \sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

If it passes through the point $(4, 0)$, then

$$\frac{4}{5} \cos\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

or $\frac{4}{5} = \frac{\cos\{(\theta-\phi)/2\}}{\cos\{(\theta+\phi)/2\}}$

or $\frac{4+5}{4-5} = \frac{\cos\{(\theta-\phi)/2\} + \cos\{(\theta+\phi)/2\}}{\cos\{(\theta-\phi)/2\} - \cos\{(\theta+\phi)/2\}}$

$$= \frac{2 \cos(\theta/2) \cos(\phi/2)}{2 \sin(\phi/2) \sin(\theta/2)}$$

or $\tan \frac{\theta}{2} \tan \frac{\phi}{2} = -\frac{1}{9}$

If it passes through the point $(-4, 0)$, then $\tan \frac{\phi}{2} \tan \frac{\theta}{2} = 9$

8. (1), (3), (4)

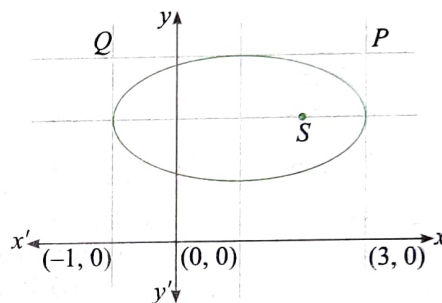
$$x^2 + 4y^2 - 2x - 16y + 13 = 0$$

or $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 4$

or $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} = 1$

$$\therefore \text{Length of latus rectum} = \frac{2 \times 1}{2} = 1$$

Also, $e = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$



or $2ae = 2 \times 2 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$

Sum of focal distance $= 2a = 4$

Tangents at the vertices are

$$x - 1 = \pm 2$$

or $x = 3, -1$

Therefore, the line $y = 3$ intersects these at points $P(3, 3)$ and $Q(-1, 3)$.

One of the foci is $S(\sqrt{3} + 1, 2)$.

$$\text{Slope of } PS = \frac{1}{2 - \sqrt{3}}$$

$$\text{Slope of } QS = \frac{1}{-2 - \sqrt{3}}$$

$$\therefore \text{Product of slopes} = \frac{1}{2 - \sqrt{3}} \times \frac{1}{-2 - \sqrt{3}} = -1$$

9. (1), (2), (3)

(1) The given equation is

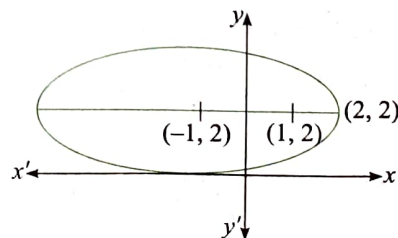
$$\left(x - \frac{1}{13}\right)^2 + \left(y - \frac{2}{13}\right)^2 = \frac{1}{a^2} \left(\frac{5x + 12y - 1}{13}\right)^2$$

It represents an ellipse if

$$\frac{1}{a^2} < 1 \text{ or } a^2 > 1 \text{ or } a > 1$$

(2) $4x^2 + 8x + 9y^2 - 36y = -4$

or $4(x^2 + 2x + 1) + 9(y^2 - 4y + 4) = 36$



$$\text{or } \frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$$

Hence, $(-1, 2)$ is center and $(1, 2)$ lies on the major axis. Then the required minimum distance is 1.

- (3) The equation of normal at $P(\theta)$ is $5 \sec \theta x - 4 \operatorname{cosec} \theta y = 25 - 16$, and it passes through $P(0, \alpha)$. Therefore,

$$\alpha = \frac{-9}{4 \operatorname{cosec} \theta} = \frac{-9}{4} \sin \theta$$

$$\text{or } |\alpha| < \frac{9}{4}$$

$$(4) \frac{2b^2}{a} = \frac{2a}{3}$$

$$\text{or } 3b^2 = a^2$$

From $b^2 = a^2(1 - e^2)$, we have $1 = 3(1 - e^2)$ or $e = \sqrt{2/3}$.

10. (1), (3)

Let $P(h, k)$ be the point of intersection of E_1 and E_2 . Then,

$$\frac{h^2}{a^2} + k^2 = 1$$

$$\text{or } h^2 = a^2(1 - k^2) \quad (1)$$

$$\text{and } \frac{h^2}{1} + \frac{k^2}{a^2} = 1$$

$$\text{or } k^2 = a^2(1 - h^2) \quad (2)$$

Eliminating a from (1) and (2), we get

$$\frac{h^2}{1 - k^2} = \frac{k^2}{1 - h^2}$$

$$\text{or } h^2(1 - h^2) = k^2(1 - k^2)$$

$$\text{or } (h - k)(h + k)(h^2 + k^2 - 1) = 0$$

Hence, the locus is a set of curves consisting of the straight lines $y = x$, $y = -x$, and the circle $x^2 + y^2 = 1$.

11. (1), (3)

$f(x)$ is a decreasing function and for the major axis to be the x -axis,

$$f(k^2 + 2k + 5) > f(k + 11)$$

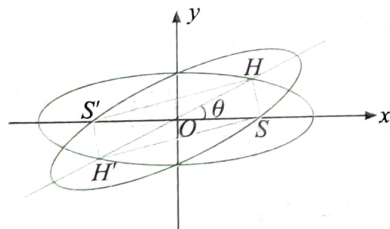
$$\text{or } k^2 + 2k + 5 < k + 11$$

$$\text{or } k \in (-3, 2)$$

Then for the remaining values of k , i.e., $k \in (-\infty, -3) \cup (2, \infty)$, the major axis is the y -axis.

12. (1), (2), (3)

Clearly, O is the midpoint of SS' and HH' .



Therefore, the diagonals of quadrilateral $HS'H'S$ bisect each other. So it is a parallelogram.

Let $H'O = 2r$. Then, $OH = r = ae'$.

The point H lies on

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ (Suppose)}$$

$$\therefore \frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} = 1$$

$$e^2 \cos^2 \theta + \frac{e^2 \sin^2 \theta}{1 - e^2} = 1$$

$$[\because b^2 = a^2(1 - e^2)]$$

$$\text{or } e^2 \cos^2 \theta - \frac{e^2 \cos^2 \theta}{1 - e^2} = 1 - \frac{e^2}{1 - e^2}$$

$$\text{or } \cos^2 \theta = \frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2 e'^2}$$

$$\text{or } \theta = \cos^{-1} \sqrt{\frac{1}{e^2} + \frac{1}{e'^2} - \frac{1}{e^2 e'^2}}$$

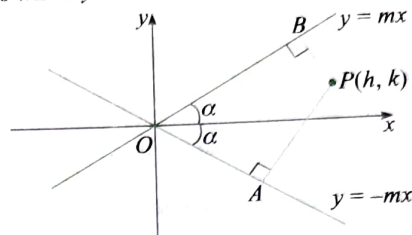
For $\theta = 90^\circ$,

$$\frac{e^2 + e'^2}{e^2 e'^2} = \frac{1}{e^2 e'^2}$$

$$\text{or } e^2 + e'^2 = 1$$

13. (3), (4)

Let us choose the point of intersection of the given lines as the origin and their angular bisector as the x -axis. The equation of the two lines will be $y = mx$ and $y = -mx$, where $m = \tan \alpha$.



Let $P(h, k)$ be the point whose locus is to be found. Then according to the given condition,

$$PA^2 + PB^2 = \text{Constant}$$

$$\Rightarrow \frac{(k - mh)^2}{1 + m^2} + \frac{(k + mh)^2}{1 + m^2} = c \quad (c \text{ is a constant})$$

$$\Rightarrow 2(k^2 + m^2 h^2) = c(1 + m^2)$$

Therefore, the locus of point P is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\text{where } a^2 = \frac{c(1 + m^2)}{2m^2} \text{ and } b^2 = \frac{c(1 + m^2)}{2}$$

If $\alpha < \pi/4$, then $m < 1$, and $a^2 > b^2$.

$$\therefore \text{Eccentricity} = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - m^2} = \frac{\sqrt{\cos 2\alpha}}{\cos \alpha}$$

If $\alpha > \pi/4$, then $m > 1$, and $a^2 < b^2$.

$$\therefore \text{Eccentricity} = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{1}{m^2}} = \frac{\sqrt{-\cos 2\alpha}}{\sin \alpha}$$

14. (1), (3)

Tangent and normal are bisectors of $\angle SPS'$.

Now, the equation of SP is $y = 3x/2$ and that of $S'P$ is $y = 5x$.

Then equations of angle bisectors are

$$\frac{3x - 2y}{\sqrt{13}} = \pm \frac{5x - y}{\sqrt{26}}$$

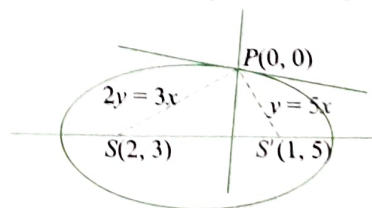
$$\text{or } 3x - 2y = \pm \frac{5x - y}{\sqrt{2}}$$

Therefore, the lines are

$$(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$$

$$\text{and } (3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$$

Now, $(2, 3)$ and $(1, 5)$ lie on the same side of $(3\sqrt{2} - 5)x + (1 - 2\sqrt{2})y = 0$, which is the equation of tangent.



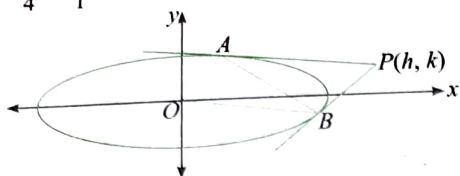
Points $(2, 3)$ and $(1, 5)$ lie on the different sides of $(3\sqrt{2} + 5)x - (2\sqrt{2} + 1)y = 0$, which is the equation of normal.

15. (1), (3)

Let the point of intersection of tangents A and B be $P(h, k)$. Then the equation of AB is

$$\frac{xh}{4} + \frac{yk}{1} = 1$$

(1)



Homogenizing the equation of ellipse using (1), we get

$$\frac{x^2}{4} + \frac{y^2}{1} = \left(\frac{xh}{4} + \frac{yk}{1}\right)^2$$

$$\text{or } x^2 \left(\frac{h^2 - 4}{16}\right) + y^2(k^2 - 1) + \frac{2hk}{4}xy = 0 \quad (2)$$

The given equation of OA and OB is

$$x^2 + 4y^2 + \alpha xy = 0 \quad (3)$$

Since (2) and (3) represent the same line,

$$\frac{h^2 - 4}{16} = \frac{k^2 - 1}{4} = \frac{hk}{2a}$$

$$\text{or } h^2 - 4 = 4(k^2 - 1)$$

$$\text{or } h^2 - 4k^2 = 0$$

Therefore, the locus is $(x - 2y)(x + 2y) = 0$.

16. (2), (4)

Differentiating the equation of ellipse, $x^2 + 3y^2 = 37$, w.r.t. x , we get

$$\frac{dy}{dx} = -\frac{x}{3y}$$

The slope of the given line is $6/5$, which is normal to the ellipse.

Hence, $3y/x = 6/5$ or $2x = 5y$.

Points in options (2) and (4) are satisfying the above equation and that of ellipse.

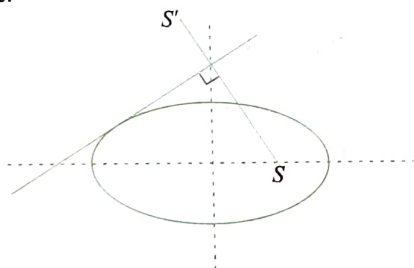
17. (1), (3)

Let $S'(h, k)$ be the image.

SS' cuts a tangent at a point which lies on the auxiliary circle of the ellipse. Therefore,

$$\left(\frac{h \pm 4}{2}\right)^2 + \frac{k^2}{4} = 25$$

Hence, the locus is $(x \pm 4)^2 + y^2 = 100$.



18. (1), (4)

The equation of the tangent at $(t^2, 2t)$ to the parabola $y^2 = 4x$ is

$$2ty = 2(x + t^2)$$

$$\text{or } ty = x + t^2$$

$$\text{or } x - ty + t^2 = 0 \quad (1)$$

The equation of the normal at $(\sqrt{5} \cos \theta, 2 \sin \theta)$ on the ellipse $4x^2 + 5y^2 = 20$ is

$$(\sqrt{5} \sec \theta)x - (2 \operatorname{cosec} \theta)y = 5 - 4$$

$$\text{or } (\sqrt{5} \sec \theta)x - (2 \operatorname{cosec} \theta)y = 1 \quad (2)$$

Given that (1) and (2) represent the same line. Then,

$$\frac{\sqrt{5} \sec \theta}{1} = \frac{-2 \operatorname{cosec} \theta}{-t} = \frac{-1}{t^2}$$

$$\text{or } t = \frac{2}{\sqrt{5}} \cot \theta \text{ and } t = -\frac{1}{2} \sin \theta$$

$$\text{or } \frac{2}{\sqrt{5}} \cot \theta = -\frac{1}{2} \sin \theta$$

$$\text{or } 4 \cos \theta = -\sqrt{5} \sin^2 \theta$$

$$\text{or } 4 \cos \theta = -\sqrt{5}(1 - \cos^2 \theta)$$

$$\text{or } \sqrt{5} \cos^2 \theta - 4 \cos \theta - \sqrt{5} = 0$$

$$\text{or } (\cos \theta - \sqrt{5})(\sqrt{5} \cos \theta + 1) = 0$$

$$\text{or } \cos \theta = -\frac{1}{\sqrt{5}} \quad [\because \cos \theta \neq -\sqrt{5}]$$

$$\text{or } \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$$

Putting $\cos \theta = -1/\sqrt{5}$ in $t = -(1/2) \sin \theta$, we get

$$t = -\frac{1}{2} \sqrt{1 - \frac{1}{5}} = -\frac{1}{\sqrt{5}}$$

$$\text{Hence, } \theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right) \text{ and } t = -\frac{1}{\sqrt{5}}$$

19. (1), (4)

Equation of the normal at $(a \cos \theta, b \sin \theta)$ is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

Comparing with $4x - 3y = 7$, we get

$$\frac{a \sec \theta}{4} = \frac{b \operatorname{cosec} \theta}{3} = \frac{a^2 - b^2}{7} \quad (1)$$

For $a > b$,

$$b^2 = a^2(1 - e^2)$$

$$\therefore b = \frac{3a}{4} \quad (2)$$

From equation (1), we get

$$\frac{\frac{a}{4}}{\cos \theta} = \frac{\frac{b}{3}}{\sin \theta} = \sqrt{\frac{a^2}{16} + \frac{b^2}{9}} = \frac{a^2 - b^2}{7} \quad (3)$$

Solving (2) and (3), we get

$$\Rightarrow a = 4\sqrt{2}, b = 3\sqrt{2}$$

$$L.R. = \frac{2b^2}{a} = \frac{2 \times 18}{4\sqrt{2}} = \frac{9}{\sqrt{2}}$$

For $a < b$, we have

$$a^2 = b^2(1 - e^2)$$

$$\Rightarrow a = \frac{3b}{4}$$

$$\Rightarrow -7 \sqrt{\frac{9}{16 \times 16} b^2 + \frac{b^2}{9}} = \frac{9}{16} b^2 - b^2 \quad (\text{Using (3)})$$

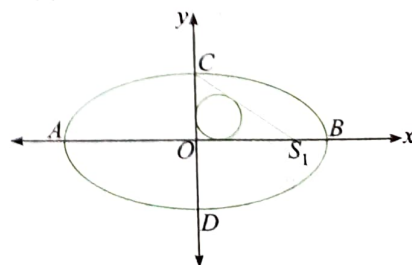
$$\Rightarrow \frac{\sqrt{337}}{3 \times 16} = \frac{b}{16}$$

$$\therefore b = \frac{\sqrt{337}}{3}, a = \frac{\sqrt{337}}{4}$$

$$L.R. = \frac{2a^2}{b} = \frac{2 \times \frac{337}{16}}{\frac{\sqrt{337}}{3}} = \frac{3\sqrt{337}}{8}$$

Linked Comprehension Type

1. (3), 2. (1), 3. (1)



$$\therefore OS_1 = ae = 6, OC = b$$

$$\text{also, } CS_1 = a$$

$$\therefore \text{Area of } \triangle OCS_1 = \frac{1}{2} (OS_1) \times (OC) = 3b$$

$$\therefore \text{Semi-perimeter of } \triangle OCS_1 = \frac{1}{2} (OS_1 + OC + CS_1)$$

$$= \frac{1}{2} (6 + a + b) \quad (1)$$

$$\therefore \text{Inradius of } \triangle OCS_1 = 1$$

$$\therefore \frac{3b}{\frac{1}{2}(6+a+b)} = 1 \quad \left(\text{Using } r = \frac{\Delta}{S} \right)$$

$$\text{or } 5b = 6 + a \quad (2)$$

$$\text{Also, } b^2 = a^2 - a^2 e^2 = a^2 - 36 \quad (3)$$

From (2), we get

$$25(a^2 - 36) = 36 + a^2 + 12a$$

$$\text{or } 2a^2 - a - 78 = 0$$

$$\text{or } a = \frac{13}{2}, -6$$

$$\therefore a = \frac{13}{2} \text{ and } b = \frac{5}{2}$$

$$\text{Area of ellipse} = \pi ab = \frac{65\pi}{4} \text{ sq. unit}$$

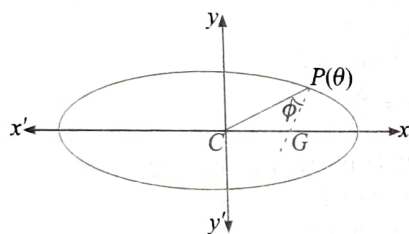
$$\text{Perimeter of } \triangle OCS_1 = 6 + a + b = 6 + \frac{13}{2} + \frac{5}{2} = 15 \text{ units}$$

The equation of director circle is

$$x^2 + y^2 = a^2 + b^2$$

$$\text{or } x^2 + y^2 = \frac{97}{2}$$

4. (3), 5. (2), 6. (1)



Let P be any point on the ellipse at $(a \cos \theta, b \sin \theta)$.

Therefore, the equation of CP is

$$y = \left(\frac{b}{a} \tan \theta \right) x$$

The normal to the ellipse at P is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$$

Slopes of the lines CP and the normal GP are $(b/a) \tan \theta$ and $(a/b) \tan \theta$, respectively. Therefore,

$$\tan \phi = \frac{\frac{a}{b} \tan \theta - \frac{b}{a} \tan \theta}{1 + \frac{a}{b} \tan \theta \frac{b}{a} \tan \theta}$$

$$= \frac{a^2 - b^2}{ab} \frac{\tan \theta}{\sec^2 \theta}$$

$$= \frac{a^2 - b^2}{ab} \sin \theta \cos \theta = \frac{a^2 - b^2}{2ab} \sin 2\theta$$

Therefore, the greatest value of $\tan \phi$ is

$$\frac{a^2 - b^2}{2ab} \times 1 = \frac{a^2 - b^2}{2ab}$$

Given that

$$\frac{a^2 - b^2}{2ab} = \frac{3}{2}$$

$$\text{Let } \frac{a}{b} = t$$

$$\therefore t - \frac{1}{t} = \frac{3}{2}$$

$$\text{or } 2t^2 - 3t - 2 = 0$$

$$\text{or } 2t^2 - 4t + t - 2 = 0$$

$$\text{or } (2t + 1)(t - 2) = 0$$

$$\text{or } \frac{a}{b} = 2$$

$$\text{or } e^2 = 1 - \frac{1}{4}$$

$$\text{or } e = \frac{\sqrt{3}}{2}$$

The rectangle inscribed in the ellipse, whose one vertex is $(a \cos \theta, b \sin \theta)$, is $(2a \cos \theta)(2b \sin \theta) = 2ab \sin(2\theta)$ which has maximum value $2ab$. Given that $a = 10$. Then $b = 5$. Therefore, the maximum area is 100.

The locus of the intersection point of perpendicular tangents is $x^2 + y^2 = 10^2 + 5^2$ or $x^2 + y^2 = 125$ (director circle).

7. (2), 8. (1), 9. (3)

$$21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$$

$$3(x - 3y + 3)^2 + 2(3x + y - 1)^2 = 180$$

$$\text{or } \frac{(x - 3y + 3)^2}{60} + \frac{(3x + y - 1)^2}{90} = 1$$

$$\text{or } \left(\frac{x - 3y + 3}{\sqrt{1 + 3^2} \sqrt{6}} \right)^2 + \left(\frac{3x + y - 1}{3\sqrt{1 + 3^2}} \right)^2 = 1$$

Thus, C is an ellipse whose lengths of axes are $6, 2\sqrt{6}$.

The minor and the major axes are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$, respectively.

Their point of intersection gives the center of the conic. Therefore,

Center $\equiv (0, 1)$

10. (2), 11. (4), 12. (1)

Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The line $y = mx \pm \sqrt{a^2 m^2 + b^2}$ touches the ellipse for all m .

Hence, it is identical with

$$y = -\frac{2px}{\sqrt{1 - p^2}} + \frac{1}{\sqrt{1 - p^2}}$$

$$\text{Hence, } m = -\frac{2p}{\sqrt{1 - p^2}}$$

$$\text{and } a^2 m^2 + b^2 = \frac{1}{1 - p^2}$$

$$\text{or } a^2 \frac{4p^2}{1 - p^2} + b^2 = \frac{1}{1 - p^2}$$

$$\text{or } p^2(4a^2 - b^2) + b^2 - 1 = 0$$

This equation is true for all real p if

$$b^2 = 1 \text{ and } 4a^2 = b^2$$

$$\text{or } b^2 = 1 \text{ and } a^2 = \frac{1}{4}$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{1/4} + \frac{y^2}{1} = 1$$

If e is its eccentricity, then

$$\frac{1}{4} = 1 - e^2 \text{ or } e^2 = \frac{3}{4} \text{ or } e = \frac{\sqrt{3}}{2}$$

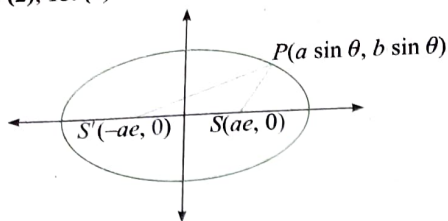
$$be = \frac{\sqrt{3}}{2}$$

Hence, the foci are $(0, \pm\sqrt{3}/2)$.

The equation of director circle is

$$x^2 + y^2 = \frac{5}{4}$$

13. (1), 14. (2), 15. (1)



Let the coordinates of P be $(a \cos \theta, b \sin \theta)$.

Here, SP = Focal distance of point $P = a - ae \cos \theta$

$$S'P = a + ae \cos \theta$$

$$SS' = 2ae$$

If (h, k) are the coordinates of the incenter of $\Delta PSS'$, then

$$h = \frac{2ae(a \cos \theta) + a(1 - e \cos \theta)(-ae) + a(1 + e \cos \theta)ae}{2ae + a(1 - e \cos \theta) + a(1 + e \cos \theta)}$$

$$= ae \cos \theta \quad (1)$$

$$\text{and } k = \frac{2ae(b \sin \theta) + a(1 - e \cos \theta) \times 0 + a(1 + e \cos \theta) \times 0}{2ae + a(1 - e \cos \theta) + a(1 + e \cos \theta)}$$

$$= \frac{eb \sin \theta}{(e + 1)} \quad (2)$$

Eliminating θ from (1) and (2), we get

$$\frac{x^2}{a^2 e^2} + \frac{y^2}{\{be/(e+1)\}^2} = 1$$

which clearly represents an ellipse. Let e_1 be its eccentricity. Then

$$\frac{b^2 e^2}{(e+1)^2} = a^2 e^2 (1 - e_1^2)$$

$$\text{or } e_1^2 = 1 - \frac{b^2}{a^2 (e+1)^2}$$

$$\text{or } e_1^2 = 1 - \frac{1 - e^2}{(e+1)^2} = 1 - \frac{1 - e}{1 + e}$$

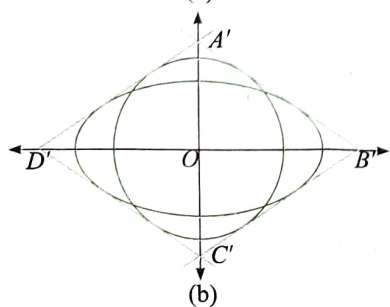
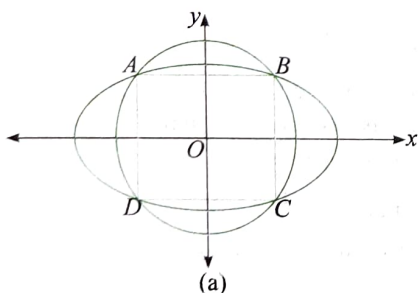
$$\text{or } e_1^2 = \frac{2e}{e+1}$$

$$\text{or } e_1 = \sqrt{\frac{2e}{e+1}}$$

Maximum area of rectangle is

$$2(ae) \left(\frac{be}{e+1} \right) = \frac{2abe^2}{e+1}$$

16. (1), 17. (4), 18. (3)



16. (1) Solving the curves given (eliminating x^2), we have

$$\frac{r^2 - y^2}{16} + \frac{y^2}{9} = 1$$

$$\text{or } y^2 = \frac{144 - 9r^2}{7}$$

Solving the curves given (eliminating y^2), we have

$$\frac{x^2}{16} + \frac{r^2 - x^2}{9} = 1$$

$$\text{or } x^2 = \frac{16r^2 - 144}{7}$$

If $ABCD$ is a square, then

$$x^2 = y^2$$

$$\text{or } \frac{144 - 9r^2}{7} = \frac{16r^2 - 144}{7}$$

$$\text{or } 25r^2 = 288$$

$$\text{or } r = \frac{12}{5}\sqrt{2}$$

17. (4) Tangents of slope m to the circle and ellipse are, respectively,

$$y = mx \pm \sqrt{r^2 m^2 + r^2}$$

$$\text{and } y = mx \pm \sqrt{16m^2 + 9}$$

For common tangent,

$$r^2 m^2 + r^2 = 16m^2 + 9$$

Also, if A', B', C' , and D' is a square, then

$$m = \pm 1$$

$$\text{or } r^2 + r^2 = 25$$

$$\text{or } r = \frac{5}{\sqrt{2}}$$

18. (3) If $A'B'C'D'$ is a square, then tangents are

$$y = \pm x \pm 5$$

for which diagonal length $A'C'$ is 10.

Then the area of circumcircle of $\Delta A'B'C'$ is 25π .

Also, the area of circle C_1 is $25\pi/2$. Hence, the required ratio is $1/2$.

19. (3), 20. (2), 21. (4)

19. (3) $\lambda x - y + 2(1 + \lambda) = 0$

$$\text{or } \lambda(x + 2) - (y - 2) = 0, \lambda \in \mathbb{R}$$

This line passes through $(-2, 2)$.

$$\mu x - y + 2(1 - \mu) = 0, \mu \in \mathbb{R}$$

$$\text{or } \mu(x - 2) - (y - 2) = 0$$

This line passes through $(2, 2)$.

Clearly, $(-2, 2)$ and $(2, 2)$ represent the foci of the ellipse. So, $2ae = 4$.

The circle $x^2 + y^2 - 4y - 5 = 0$, i.e., $x^2 + (y - 2)^2 = 9$ represents an auxiliary circle. Thus, $a^2 = 9$ or $e = 2/3$ and $b^2 = 5$.

20. (2) Required area = $\frac{1}{2} (2ae)(b) = 3 \times \frac{2}{3} \times \sqrt{5} = 2\sqrt{5}$

22. (2), 23. (4), 24. (3)

Tangent at $P(4 \cos \theta, 4 \sin \theta)$ to $x^2 + y^2 = 16$ is

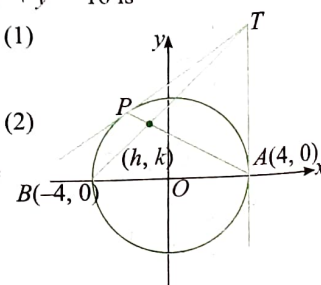
$$x \cos \theta + y \sin \theta = 4 \quad (1)$$

The equation of AP is

$$y = \frac{\sin \theta}{\cos \theta - 1} (x - 4) \quad (2)$$

From (1), the coordinates of the point T are given by

$$\left(4, \frac{4(1 - \cos \theta)}{\sin \theta} \right)$$



The equation of BT is

$$y = \frac{1 - \cos \theta}{2 \sin \theta} (x + 4) \quad (3)$$

Let (h, k) be the point of intersection of the lines (2) and (3). Then we have

$$k^2 = -\frac{1}{2}(h^2 - 16)$$

$$\text{or } \frac{h^2}{16} + \frac{k^2}{8} = 1$$

Therefore, the locus of (h, k) is

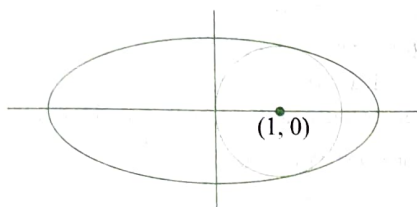
$$\frac{x^2}{16} + \frac{y^2}{8} = 1$$

which is an ellipse with eccentricity $e = 1/\sqrt{2}$.

Sum of focal distances of any point is $2a = 8$.

Considering the circle $x^2 + y^2 = a^2$, we find that the eccentricity of the ellipse is $1/\sqrt{2}$ which is constant and does not change by changing the radius of the circle.

25. (1), 26. (3), 27. (2)



Solving both equations, we have

$$\frac{x^2}{a^2} + \frac{1 - (x - 1)^2}{b^2} = 1$$

$$\text{or } b^2 x^2 + a^2 [1 - (x - 1)^2] = a^2 b^2$$

$$\text{or } (b^2 - a^2)x^2 + 2a^2x - a^2b^2 = 0 \quad (1)$$

For least area, the circle must touch the ellipse.

Therefore, the discriminant of (1) is zero. Thus,

$$4a^4 + 4a^2b^2(b^2 - a^2) = 0$$

$$\text{or } a^2 + b^2(b^2 - a^2) = 0$$

$$\text{or } a^2 + b^2(-a^2e^2) = 0$$

$$\text{or } 1 - b^2e^2 = 0$$

$$\text{or } b = \frac{1}{e}$$

$$\text{Also, } a^2 = \frac{b^2}{1 - e^2} = \frac{1}{e^2(1 - e^2)}$$

$$\text{or } a = \frac{1}{e\sqrt{1 - e^2}}$$

Let S be the area of the ellipse. Then,

$$S = \pi ab = \frac{\pi}{e^2\sqrt{1 - e^2}} = \frac{\pi}{\sqrt{e^4 - e^6}}$$

The area is minimum if $f(e) = e^4 - e^6$ is maximum, i.e., when $f'(e) = 4e^3 - 6e^5 = 0$

$$\text{or } e = \sqrt{\frac{2}{3}} \text{ [which is the point of maxima for } f(e)]$$

So, S is the least when $e = \sqrt{2/3}$.

Therefore, the ellipse is $2x^2 + 6y^2 = 9$.

The equation of auxiliary circle of ellipse is $x^2 + y^2 = 4.5$.

$$\text{Length of latus rectum of ellipse} = \frac{2b^2}{a} = \frac{2 \times \frac{9}{2}}{\frac{3}{\sqrt{2}}} = \sqrt{2}$$

Matrix Match Type

1. $a \rightarrow p, q, r, s$; $b \rightarrow p, q, r, s$; $c \rightarrow q, r, s$; $d \rightarrow p$.

a. The equation of tangent at $((\cos \theta)/2, (\sin \theta)/3)$ is

$$2x \cos \theta + 3y \sin \theta = 1$$

which is parallel to the given line $8x = 9y$. Therefore,

$$\cos \theta = \pm \frac{4}{5}, \sin \theta = \mp \frac{3}{5}$$

Hence, the points are $(2/5, -1/5)$ and $(-2/5, 1/5)$.

The distance between the points is

$$\sqrt{\frac{16}{25} + \frac{4}{25}} = \frac{2}{\sqrt{5}}$$

which is less than 1.

b. The given equation is

$$\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$$

$$\text{or } e^2 = 1 - \frac{9}{25} = \frac{16}{25} \text{ or } e = \frac{4}{5}$$

Hence, the foci are $S, S' \equiv (-1, -2 \pm 4) \equiv S(-1, 2)$ and $S'(-1, -6)$.

Required sum of distances = $2 + 6 = 8$

c. The equation of normal at $(3 \cos \theta, 2 \sin \theta)$ is

$$3x \sec \theta - 2y \csc \theta = 5$$

which is parallel to the given line $2x + y = 1$. Therefore,

$$\cos \theta = \mp \frac{3}{5}, \sin \theta = \pm \frac{4}{5}$$

Hence, the points are $(-9/5, 8/5)$ and $(9/5, -8/5)$.

Required sum of distances = $\frac{16}{5}$

d. Consider any point $(t, t+2)$, $t \in \mathbb{R}$, on the line $x - y + 2 = 0$.

The chord of contact of the ellipse w.r.t. this point is

$$xt + 2y(t+2) = 2$$

$$\text{or } (4y - 2) + t(x + 2y) = 0$$

This line passes through the point of intersection of lines

$$4y - 2 = 0 \text{ and } x + 2y = 0$$

$$\therefore x = -1, y = 1/2$$

Hence, the point is $(-1, 1/2)$, whose distance from $(2, 1/2)$ is 3.

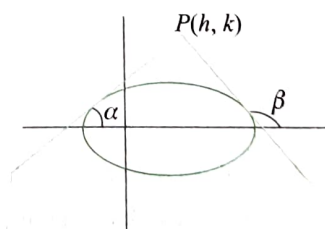
2. $a \rightarrow r, s$; $b \rightarrow p, q, r, s$; $c \rightarrow r, s$; $d \rightarrow r, s$.

The equation of any tangent to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$y = mx + \sqrt{a^2m^2 + b^2}$$



Since it passes through $P(h, k)$, we have

$$k = mh + \sqrt{a^2m^2 + b^2}$$

$$\text{or } m^2(h^2 - a^2) - 2kmh + (k^2 - b^2) = 0 \quad (2)$$

As (2) is quadratic in m , having two roots m_1 and m_2 (say), we get

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2}, m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2} \quad (3)$$

$$\text{or } \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$= \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{2hk/(h^2 - a^2)}{1 - \{(k^2 - b^2)/(h^2 - a^2)\}} = \frac{2hk}{h^2 - k^2 - a^2 + b^2}$$

$$\text{a. } \alpha + \beta = \frac{c\pi}{2}$$

When c is even,

$$m_1 + m_2 = 0$$

$$\frac{2kh}{h^2 - a^2} = 0$$

$$\text{or } 2kh = 0$$

$$\text{or } xy = 0$$

which is the equation of a pair of straight lines.

When c is odd,

$$1 - m_1 m_2 = 0$$

$$\text{or } \frac{k^2 - b^2}{h^2 - a^2} = 1$$

Therefore, the locus of (h, k) is

$$y^2 - b^2 = x^2 - a^2$$

which is a hyperbola.

$$\text{b. } m_1 m_2 = c$$

$$\text{or } \frac{k^2 - b^2}{h^2 - a^2} = c$$

When $c = 0$, $k = \pm b$, the locus is a pair of straight lines.

When $c = 1$, $h^2 - k^2 = a^2 - b^2$, the locus is a hyperbola.

When $c = -1$, $h^2 + k^2 = a^2 + b^2$, the locus is a circle.

When $c = -2$, $2h^2 + k^2 = 2a^2 + b^2$, the locus is an ellipse.

$$\text{c. } \tan \alpha + \tan \beta = c$$

$$\text{or } m_1 + m_2 = c$$

$$\text{or } \frac{2kh}{h^2 - a^2} = c$$

When $c = 0$, $kh = 0$, the locus is a pair of straight lines.

When $c \neq 0$,

$$c(h^2 - a^2) - 2kh = 0$$

The locus of (h, k) is

$$cx^2 - 2xy - ca^2 = 0$$

$$\Delta = -ca^2 \neq 0$$

$$\text{Also, } h^2 - ab = 1 > 0$$

Therefore, the locus is a hyperbola for $c \neq 0$.

$$\text{d. } \cot \alpha + \cot \beta = c$$

$$\text{or } \frac{1}{m_1} + \frac{1}{m_2} = c$$

$$\text{or } \frac{m_1 + m_2}{m_1 m_2} = c$$

$$\text{or } \frac{2kh}{k^2 - b^2} = c$$

$$\text{or } c(k^2 - b^2) - 2kh = 0$$

When $c = 0$, the locus is a pair of straight lines.

When $c \neq 0$, the locus is a hyperbola (as in previous case c).

$$3. \text{ a} \rightarrow \text{p}; \text{ b} \rightarrow \text{r}, \text{ s}; \text{ c} \rightarrow \text{p}; \text{ d} \rightarrow \text{q}.$$

a. Tangent to the ellipse at $P(\phi)$ is

$$\frac{x}{4} \cos \phi + \frac{y}{2} \sin \phi = 1$$

It must pass through the center of the circle. Hence,

$$\frac{4}{4} \cos \phi + \frac{2}{2} \sin \phi = 1$$

$$\text{or } \cos \phi + \sin \phi = 1$$

$$\text{or } 1 + \sin 2\phi = 1$$

$$\text{or } \sin 2\phi = 0$$

$$\text{i.e., } 2\phi = 0 \text{ or } \pi$$

$$\text{i.e., } \frac{\phi}{2} = 0 \text{ or } \frac{\pi}{4}$$

b. Consider any point $P(\sqrt{6} \cos \theta, \sqrt{2} \sin \theta)$ on the ellipse

$$\frac{x^2}{6} + \frac{y^2}{2} = 1$$

Given that $OP = 2$. Therefore,

$$6 \cos^2 \theta + 2 \sin^2 \theta = 4$$

$$\text{or } 4 \cos^2 \theta = 2$$

$$\text{or } \cos \theta = \pm \frac{1}{\sqrt{2}}$$

$$\text{i.e., } \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

c. Solving the equation of ellipse and parabola (eliminating x^2), we have

$$y - 1 + 4y^2 = 4$$

$$\text{or } 4y^2 + y - 5 = 0$$

$$\text{or } (4y + 5)(y - 1) = 0$$

$$\text{or } y = 1, x = 0$$

The curves touch at $(0, 1)$. so, the angle of intersection is 0.

d. The normal at $P(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{bx}{\sin \theta} = a^2 - b^2$$

where $a^2 = 14$, $b^2 = 5$.

It meets the curve again at $Q(2\theta)$, i.e., $(a \cos 2\theta, b \sin 2\theta)$.

Hence,

$$\frac{a}{\cos \theta} a \cos 2\theta - \frac{b}{\sin \theta} (b \sin 2\theta) = a^2 - b^2$$

$$\text{or } \frac{14}{\cos \theta} \cos 2\theta - \frac{5}{\sin \theta} (\sin 2\theta) = 14 - 5$$

$$\text{or } 28 \cos^2 \theta - 14 - 10 \cos^2 \theta = 9 \cos \theta$$

$$\text{or } 18 \cos^2 \theta - 9 \cos \theta - 14 = 0$$

$$\text{or } (6 \cos \theta - 7)(3 \cos \theta + 2) = 0$$

$$\text{or } \cos \theta = -\frac{2}{3}$$

$$4. \text{ a} \rightarrow \text{q}; \text{ b} \rightarrow \text{r}; \text{ c} \rightarrow \text{r}; \text{ d} \rightarrow \text{p}.$$

a. The points are $O(0, 0)$, $P(3, 4)$, and $Q(6, 8)$. Then

$$2a = OP + OQ$$

$$= 5 + 10 = 15$$

$$\text{or } a = \frac{15}{2}$$

Also, the distance between the foci,

$$2ae = \sqrt{(6-3)^2 + (8-4)^2} = 5$$

$$\text{or } e = \frac{1}{3}$$

$$\text{or } b^2 = \frac{225}{4} \left(1 - \frac{1}{9}\right) = 50$$

$$\text{or } b = 5\sqrt{2}$$

$$\text{or } 2b = 10\sqrt{2}$$

b. We know that

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$$

$$\text{or } \frac{1}{2} + \frac{1}{SQ} = \frac{10}{16}$$

$$\text{or } SQ = 8$$

$$\text{or } PQ = 10$$

c. If the line $y = x + k$ touches the ellipse $9x^2 + 16y^2 = 144$, then
 $k^2 = 16(1)^2 + 9$

or $k = \pm 5$

d. The sum of the distances of a point on the ellipse from the foci is $2a = 8$.

5. (3)

a. The locus is

$$\frac{x^2}{16} + \frac{y^2}{36} = 1$$

or $e = \sqrt{1 - \frac{16}{36}} = \sqrt{\frac{20}{36}} = \frac{\sqrt{5}}{3}$

or $3e = \sqrt{5}$

b. $3(x^2 + 2x + 1) + 2(y^2 - 2y + 1) = 3 + 2 + 1$

or $\frac{(x+1)^2}{2} + \frac{(y-1)^2}{3} = 1$

or $e = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$

$\therefore a = \sqrt{3}, b = \sqrt{2}$

$\therefore \text{Area} = \frac{1}{2} \times 2\sqrt{3} \times \sqrt{2} = \sqrt{6}$

c. Eliminating θ from $x = 1 + 4 \cos \theta$ and $y = 2 + 3 \sin \theta$, we have

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$$

Hence, $a = 4$ and e is $\sqrt{7}/4$. Therefore,

$$ae = \sqrt{7}$$

Hence, the distance between the foci is $2\sqrt{7}$.

d. $\frac{x^2}{16} + \frac{y^2}{7} = 1$

$$e = \sqrt{1 - \frac{7}{16}} = \frac{3}{4}$$

One end of the latus rectum is

$$\left(ae, \frac{b^2}{a}\right) = \left(3, \frac{7}{4}\right)$$

Therefore, the equation of tangent is

$$\frac{3x}{16} + \frac{7y}{4} = 1$$

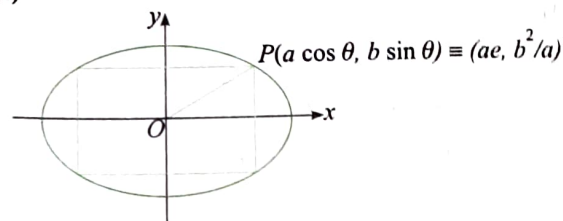
or $\frac{3x}{16} + \frac{y}{4} = 1$

It meets the x -axis at $(16/3, 0)$ and the y -axis at $(0, 4)$.

Hence, the area of quadrilateral is

$$2 \times \frac{16}{3} \times 4 = \frac{128}{3}$$

6. (4)



a. Let one of the vertices of the rectangle be $P(a \cos \theta, b \sin \theta)$. Then its area is

$$A = (2a \cos \theta)(2b \sin \theta) = 2ab \sin 2\theta$$

Hence, $A_{\max} = 2ab$

Now, the area of rectangle formed by the extremities of the latus rectum is

$$2ae \times \frac{2b^2}{a} = 4eb^2$$

Given that

$$2ab = 4eb^2$$

or $\frac{2b}{a} e = 1$

or $\frac{4b^2}{a^2} e^2 = 1$

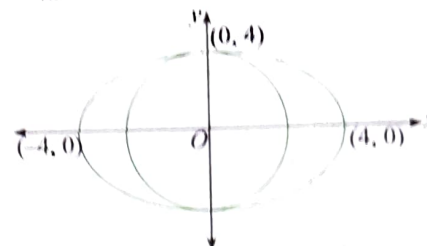
or $4(1 - e^2)e^2 = 1$

or $4e^4 - 4e^2 + 1 = 0$

or $(2e^2 - 1)^2 = 0$

or $e = \frac{1}{\sqrt{2}}$

b.



For the ellipse, the distance c between the foci, $2ae = 8$, and the length of the semi-minor axis is $b = 4$. Now,

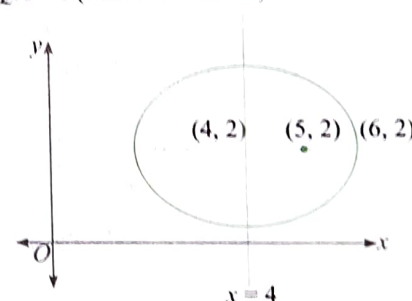
$$b^2 = a^2 - a^2 e^2$$

or $16 = a^2 - 16$

or $a^2 = 32$

or $e = \sqrt{1 - \frac{16}{32}} = \frac{1}{\sqrt{2}}$

c. Normal at point $P(6, 2)$ to the ellipse passes through its focus $Q(5, 2)$. Then P must be the extremity of the major axis. Now, $ae = QR = 1$ (where R is center) and $a - ae = 1$.

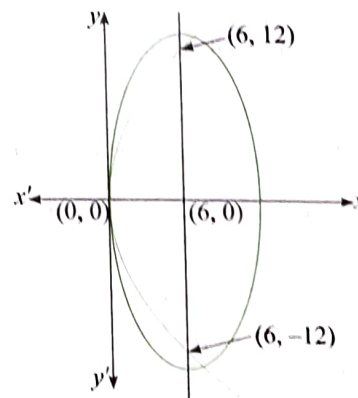


$\therefore a = 2$

$$b^2 = a^2 - a^2 e^2 = 4 - 1 = 3$$

or $e = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

d.



The extremities of the latus rectum of the parabola $y^2 = 24x$ are $(6, \pm 12)$.

For ellipse, $2be = 24$ and the extremity of the minor axis is $(0, 0)$. Hence, $a = 6$. Now,

$$a^2 = b^2 - b^2 e^2$$

or $b^2 = 36 + 144 = 180$

or $e^2 = \sqrt{1 - \frac{36}{180}} = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$

$$\begin{aligned} E &= \frac{x^2}{4} + \frac{y^2}{9} = \frac{(2\cos\theta + 4)^2}{4} + \sin^2\theta \\ &= \frac{4\cos^2\theta + 16 + 16\cos\theta + 4\sin^2\theta}{4} \\ &= \frac{20 + 16\cos\theta}{4} = 5 + 4\cos\theta \end{aligned}$$
$$a^2 e^2 = -2 \left(\frac{b^2}{a} - 2 \right) \quad (1)$$
$$e^2 = 1 + \frac{4}{3} \quad \text{or} \quad e = \sqrt{\frac{7}{3}}$$

Hence, $b^2 = 54$.

A diagram of an ellipse centered at the origin O of a Cartesian coordinate system. The major axis lies along the x -axis, with vertices A and B at $(a, 0)$ and $(-a, 0)$ respectively. The minor axis lies along the y -axis, with vertices C and D at $(0, b)$ and $(0, -b)$ respectively. A point P is located in the first quadrant on the ellipse. Line segments connect P to the vertices A , B , and C . These segments are labeled a , b , and c respectively. A dashed line segment connects P to the origin O and is labeled d .

(focal property)

$$x = 2 + \cos \theta \text{ and } y = -\frac{1}{3} + \frac{1}{3} \sin \theta$$

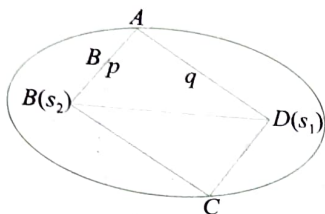
$$\therefore 4x - 9y = 4(2 + \cos \theta) - 9\left(-\frac{1}{3} + \frac{1}{3}\sin \theta\right)$$

$$\text{or } f(\theta) = 8 + 4 \cos \theta + 3 - 3 \sin \theta \\ = 11 + 4 \cos \theta - 3 \sin \theta$$

$$\therefore f(\theta)_{\max} = 11 + 5 = 16$$

11. (80) Let the sides of rectangle be p and q .

$$\text{Area of rectangle} = pq = 200$$



$$\text{Area of ellipse} = \pi ab = 200\pi$$

$$\therefore ab = 200$$

$$\text{Perimeter of rectangle} = 2(p + q)$$

From triangle ABD,

$$\text{Distance } BD = \sqrt{p^2 + q^2} = \text{Distance between foci}$$

$$\text{or } p^2 + q^2 = 4a^2e^2$$

$$\text{or } (p + q)^2 - 2pq = 4(a^2 - b^2)$$

Also, from the definition of ellipse, the sum of focal lengths is $2a$. Then,

$$AB + AD = p + q = 2a$$

Putting the value of $(p + q)$ in (3) from (4), we have

$$(2a)^2 - 2pq = 4a^2 - 4b^2$$

$$\text{or } 4a^2 - 2 \times 200 = 4(a^2 - b^2)$$

$$\text{or } a^2 - 100 = a^2 - b^2$$

$$\text{or } b = 10$$

From (2),

$$ab = 200 \text{ or } a = 20$$

$$\text{Since } p + q = 2a$$

$$\text{Perimeter} = 2(p + q) = 4a = 4 \times 20 = 80$$

(1)

(2)

(3)

(4)

[Using (1)]

[From (4)]

2. (2) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Which will pass through $(-3, 1)$ if $\frac{9}{a^2} + \frac{1}{b^2} = 1$

$$\text{And eccentricity } e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{\frac{2}{5}}$$

$$\Rightarrow \frac{2}{5} = 1 - \frac{b^2}{a^2} \Rightarrow \frac{b^2}{a^2} = \frac{3}{5} \Rightarrow b^2 = \frac{3}{5}a^2$$

$$\text{Thus } \frac{9}{a^2} + \frac{1}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{5}{3a^2} = 1$$

$$\Rightarrow a^2 = \frac{32}{3}, b^2 = \frac{3}{5} \times \frac{32}{3} = \frac{32}{5}$$

Required equation of the ellipse is $3x^2 + 5y^2 = 32$

3. (2) Equation of tangent to the ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$ is

$$y = mx \pm \sqrt{2m^2 + 4}$$

Equation of tangent to the parabola $y^2 = 16\sqrt{3}x$ is

$$y = mx + \frac{4\sqrt{3}}{m}$$

On comparing (1) and (2)

$$\Rightarrow 48 = m^2(2m^2 + 4) \Rightarrow m^4 + 2m^2 - 24 = 0$$

$$\Rightarrow (m^2 + 6)(m^2 - 4) = 0 \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$$

So, equation of common tangents are

$$y = \pm 2x \pm 2\sqrt{3}$$

Statement 1 is true.

Statement 2 is obviously true and correct explanation of statement 1.

4. (4) Length of semi-minor axis is 2.

Length of semi-major axis is 4.

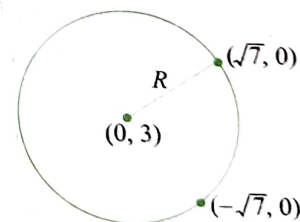
Then equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \Rightarrow x^2 + 4y^2 = 16$$

5. (1) Coordinates of foci are $(ae, 0)$; $(-ae, 0)$

$$\therefore e = \sqrt{1 - \frac{9}{16}} \Rightarrow e = \frac{\sqrt{7}}{4}$$

Coordinates of foci are $(\sqrt{7}, 0)$, $(-\sqrt{7}, 0)$



$$\text{Radius, } R = \sqrt{7 + 9} = 4$$

Thus, equation of circle is

$$(x - 0)^2 + (y - 3)^2 = 4^2 \text{ or } x^2 + y^2 - 6y + 9 = 16$$

$$\text{or } x^2 + y^2 - 6y - 7 = 0$$

6. (3) Let the foot of perpendicular be $P(h, k)$.

Equation of tangent with slope m passing through $P(h, k)$ is

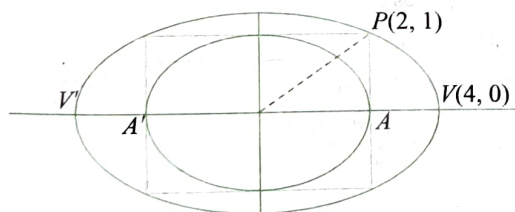
$$y = mx \pm \sqrt{6m^2 + 2}, \text{ where } m = -\frac{h}{k}$$

Archives

JEE Main

Single Correct Answer Type

1. (2)



$$x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$$

So $a = 2$, $b = 1$.

Thus P is $(2, 1)$.

The required ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{4^2} + \frac{y^2}{b^2} = 1$

The point $(2, 1)$ lies on it. So

$$\frac{4}{16} + \frac{1}{b^2} = 1 \Rightarrow \frac{1}{b^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow b^2 = \frac{4}{3}$$

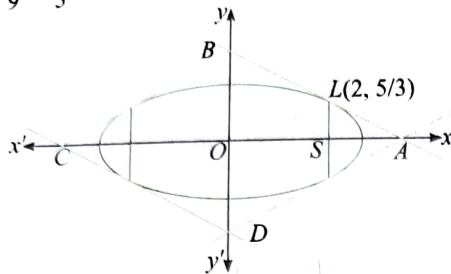
$$\therefore \frac{x^2}{16} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1 \Rightarrow \frac{x^2}{16} + \frac{3y^2}{4} = 1 \Rightarrow x^2 + 12y^2 = 16$$

$$\Rightarrow \sqrt{\frac{6h^2}{k^2} + 2} = \pm \frac{h^2 + k^2}{k} \Rightarrow 6h^2 + 2k^2 = (h^2 + k^2)^2$$

So, required locus is $6x^2 + 2y^2 = (x^2 + y^2)^2$.

7. (4) The given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$



Then, $a^2 = 9$, $b^2 = 5$. Therefore,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \frac{2}{3} \quad \therefore ae = 2$$

Hence, the endpoint of latus rectum in the first quadrant is $L(ae, b^2/a) \equiv L(2, 5/3)$.

The equation of tangent at L is

$$\frac{2x}{9} + \frac{y}{3} = 1$$

the tangent meets the x-axis at $A(9/2, 0)$ and the y-axis at $B(0, 3)$. Therefore,

$$\text{Area of } \triangle OAB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$$

$$\therefore \text{Area of quadrilateral} = 4 \times \frac{27}{4} = 27$$

8. (3) Given eccentricity, $e = 1/2$

Also, from the given equation of the directrix,

$$-\frac{a}{e} = -4 \Rightarrow a = 2$$

$$\text{Now, } b^2 = a^2(1 - e^2) = 4\left(1 - \frac{1}{4}\right) = 3$$

So, equation of ellipse is $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = -\frac{3x}{4y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(1, 3/2)} = -\frac{3}{4} \times \frac{2}{3} = -\frac{1}{2}$$

$\therefore$ Slope of the normal = 2

Thus, equation of the required normal is

$$y - \frac{3}{2} = 2(x - 1) \text{ or } 4x - 2y = 1$$

9. (3) Let $a = x$ and $b = y$.

So, we have $|x - 5| < 1$ and $|y - 5| < 1$

$$|x - 5| < 1$$

$$\Rightarrow -1 < x - 5 < 1$$

$$\Rightarrow 4 < x < 6$$

Similarly, $4 < y < 6$

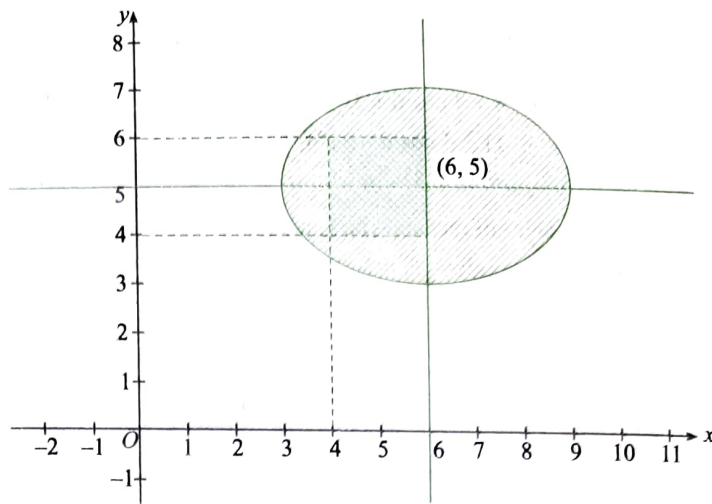
Thus, we have a square for inequality (1).

$$\text{Now, } 4(x - 6)^2 + 9(y - 5)^2 \leq 36$$

$$\Rightarrow \frac{(x - 6)^2}{9} + \frac{(y - 5)^2}{4} \leq 1 \quad (2)$$

Points satisfying above inequality lies inside ellipse or on the ellipse having center at $(6, 5)$.

Regions of both the inequalities are plotted as shown in the following figure:

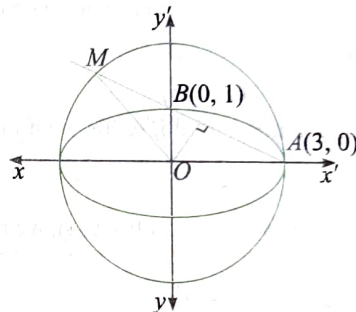


From the figure, clearly, $A \subset B$.

JEE Advanced

Single Correct Answer Type

1. (4)



The equation of line AM is $x + 3y - 3 = 0$

Perpendicular distance of line from the origin $= \frac{3}{\sqrt{10}} = OB$

$$\text{Length of } AM = 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$$

$$\therefore \text{Area of } \triangle OAM = \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10} \text{ sq. units.}$$

2. (3) Given ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Normal at $P(4 \cos \phi, 2 \sin \phi)$ is given by $4x \sec \phi - 2y \csc \phi = 12$

It meets x-axis at

$$Q \equiv (3 \cos \phi, 0)$$

Let $M \equiv (\alpha, \beta)$

be the mid point of PQ.

$$\therefore \alpha = \frac{3 \cos \phi + 4 \cos \phi}{2} = \frac{7}{2} \cos \phi$$

$$\text{or } \cos \phi = \frac{2}{7} \alpha$$

$$\text{and } \beta = \sin \phi$$

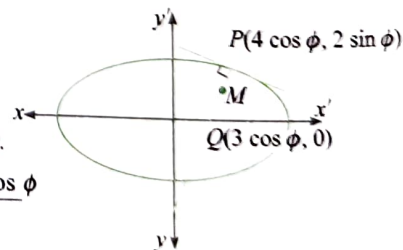
Using $\cos^2 \phi + \sin^2 \phi = 1$, we have

$$\frac{4}{49} \alpha^2 + \beta^2 = 1$$

$$\text{or } \frac{4}{49} x^2 + y^2 = 1 \quad (1)$$

Now, the latus rectum lines of given ellipse are

$$x = \pm 2\sqrt{3} \quad (2)$$



Solving (1) and (2), we have

$$\frac{48}{49} + y^2 = 1$$

$$\text{or } y = \pm \frac{1}{7}$$

The points of intersection are $(\pm 2\sqrt{3}, \pm 1/7)$.

3. (3) Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as it is passing through $(0, 4)$ and $(3, 2)$. So,

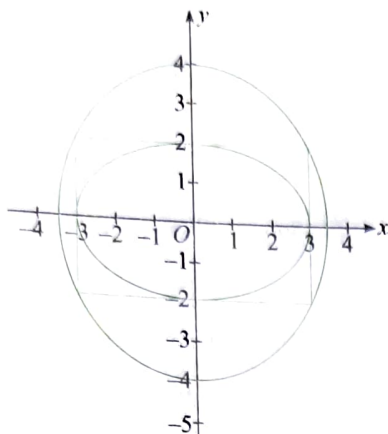
$$b^2 = 16 \text{ and}$$

$$\frac{9}{a^2} + \frac{4}{16} = 1$$

$$\text{or } a^2 = 12$$

$$\text{So, } 12 = 16(1 - e^2)$$

$$\text{or } e = \frac{1}{2}$$



Multiple Correct Answers Type

1. (2), (3)

$$\text{Given } \cos B + \cos C = 4 \sin^2 \frac{A}{2}$$

$$\text{or } 2 \cos\left(\frac{B+C}{2}\right) \cos\left(\frac{B-C}{2}\right) = 4 \sin^2 \frac{A}{2}$$

$$\text{or } \cos\left(\frac{B-C}{2}\right) = 2 \sin^2 \frac{A}{2}$$

$$\text{or } 2 \sin \frac{B+C}{2} \cos \frac{B-C}{2} = 4 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\text{or } \sin B + \sin C = 2 \sin A$$

$$\text{or } b + c = 2a$$

Thus sum of two variable sides is constant.

Hence, the locus of vertex A is an ellipse with B and C as foci.

2. (1), (2)

$$\text{Let ellipses be } E_1: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } E_2: \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

Since $x + y = 3$ is a tangent,

$$a^2 + b^2 = A^2 + B^2 = 9 \text{ (using condition } c^2 = a^2 m^2 + b^2 \text{ etc.)}$$

Point P lies on $x^2 + (y-1)^2 = 2$.

Equation of normal to circle having slope 1 is

$$y - 1 = 1(x - 0) \text{ or } x - y + 1 = 0$$

Solving this normal with tangent line we get point $P(1, 2)$.

$$\text{Now } PQ = PR = \frac{2\sqrt{2}}{3}$$

So, points on line $x + y - 3 = 0$ at distance $\frac{2\sqrt{2}}{3}$ from point

$$P \text{ are } \left(1 \mp \frac{1}{\sqrt{2}} \frac{2\sqrt{2}}{3}, 2 \mp \frac{1}{\sqrt{2}} \frac{2\sqrt{2}}{3}\right)$$

$$\text{or } Q\left(\frac{5}{3}, \frac{4}{3}\right) \text{ and } Q\left(\frac{1}{3}, \frac{8}{3}\right)$$

Now $Q\left(\frac{5}{3}, \frac{4}{3}\right)$ lies on E_1

$$\text{So, } \frac{25}{9a^2} + \frac{16}{9(9-a^2)} = 1$$

$$\Rightarrow 225 - 25a^2 + 16a^2 = 9a^2(9 - a^2)$$

$$\Rightarrow a^4 - 10a^2 + 25 = 0$$

$$\Rightarrow a^2 = 5 \text{ so } b^2 = 4$$

$$\therefore e_1^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{5} = \frac{1}{5}$$

Now $\left(\frac{1}{3}, \frac{8}{3}\right)$ lies on E_2

$$\text{So, } \frac{1}{A^2} + \frac{64}{9(9-A^2)} = 9$$

$$\Rightarrow 9 - A^2 + 64A^2 = 9A^2(9 - A^2)$$

$$\Rightarrow A^4 - 2A^2 + 1 = 0$$

$$\Rightarrow A^2 = 1 \text{ so, } B^2 = 8$$

$$\therefore e_2 = 1 - \frac{1}{8} = \frac{7}{8}$$

3. (1), (3)

Equation of tangent to parabola $y^2 = 4x$ having slope m is

$$y = mx + \frac{1}{m}$$

If this is tangent to circle $x^2 + y^2 = 1/2$, then distance from centre of the circle from the line is equal to radius.

$$\therefore \frac{\left|0 + 0 + \frac{1}{m}\right|}{\sqrt{1 + m^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore m^4 + m^2 - 2 = 0$$

$$\therefore m = \pm 1$$

Equation of common tangents are $y = x + 1$ and $y = -x - 1$.

These tangents intersect at point Q which is $(-1, 0)$.

$$\therefore OQ = 1.$$

Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

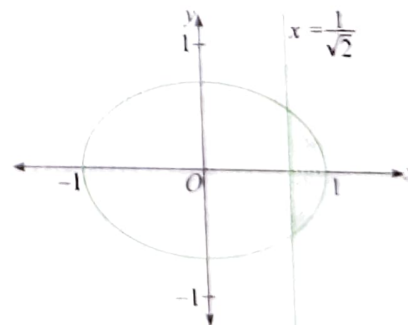
$$\therefore a = OQ = 1$$

Also, given that $2b = \sqrt{2}$ or $b = \frac{1}{\sqrt{2}}$

Therefore, equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$.

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\text{L.R.} = \frac{2b^2}{a} = 1$$



Let the area bounded by ellipse and lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ be A .

$$A = 2 \times \int_{1/\sqrt{2}}^1 \frac{1}{\sqrt{2}} \sqrt{1 - x^2} dx$$

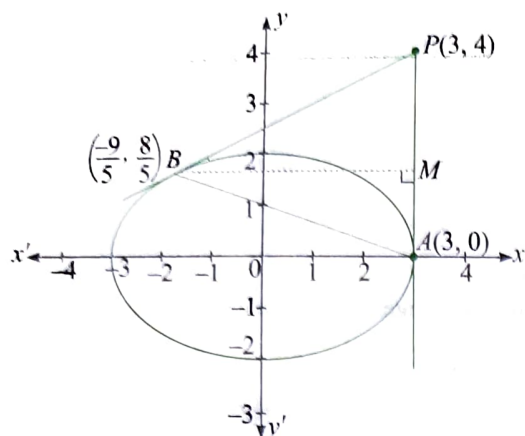
$$= \sqrt{2} \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x \right]_{1/\sqrt{2}}^1$$

$$= \sqrt{2} \left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8} \right) \right] = \sqrt{2} \left(\frac{\pi}{8} - \frac{1}{4} \right) = \frac{\pi - 2}{4\sqrt{2}}$$

Linked Comprehension Type

1. (4) Given ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

The equation of tangent having slope m is



$$y = mx + \sqrt{9m^2 + 4}$$

The tangent passes through the point (3, 4). Therefore,

$$4 - 3m = \sqrt{9m^2 + 4}$$

Squaring, we have

$$16 + 9m^2 - 24m = 9m^2 + 4$$

$$\text{or } m = \frac{12}{24} = \frac{1}{2}$$

Therefore, the equation of tangent is

$$y - 4 = -(x - 3) \text{ or } x - 2y + 5 = 0 \quad (1)$$

Let the point of contact on the curve be $B(\alpha, \beta)$.

Equation of tangent at this point is

$$\frac{x\alpha}{9} + \frac{y\beta}{4} - 1 = 0 \quad (2)$$

Comparing equations (1) and (2)

$$\text{or } \frac{\alpha/9}{1} = \frac{\beta/4}{-2} = -\frac{1}{5}$$

$$\text{or } \alpha = -\frac{9}{5}, \beta = \frac{8}{5} \quad \therefore B \equiv \left(-\frac{9}{5}, \frac{8}{5}\right)$$

Another slope of tangent is ∞ . Then the equation of tangent is $x = 3$ and the corresponding point of contact is $A(3, 0)$.

2. (3) Since the slope of PA is ∞ , the slope of altitude through B must be 0, for which the orthocenter is $(11/5, 8/5)$.

3. (1) The equation of chord of contact AB is

$$\frac{3x}{9} + \frac{4y}{4} = 1 \text{ or } \frac{x}{3} + y = 1 \text{ or } x + 3y - 3 = 0$$

Thus, equation of locus of the point whose distances from the point P and the line AB are equal is

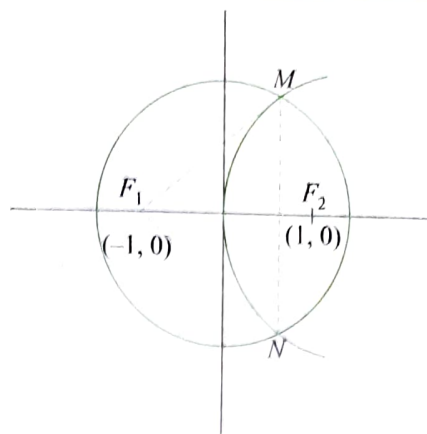
$$(x - 3)^2 + (y - 4)^2 = \frac{(x + 3y - 3)^2}{10}$$

$$\text{or } 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$$

4. (1) $a = 3, e = \frac{1}{3}$

$$\therefore F_1 \equiv (-1, 0) \text{ and } F_2 \equiv (1, 0)$$

So, equation of parabola is $y^2 = 4x$.



Solving ellipse and parabola, we get points of intersection as

$$M\left(\frac{3}{2}, \sqrt{6}\right) \text{ and } N\left(\frac{3}{2}, -\sqrt{6}\right).$$

Orthocentre of ΔF_1MN lies on x -axis.

$$\text{Slope of } F_1N = \frac{-\sqrt{6} - 0}{\frac{3}{2} - (-1)} = \frac{-2\sqrt{6}}{5}$$

Therefore, line through M and perpendicular to F_1N is:

$$y - \sqrt{6} = \frac{5}{2\sqrt{6}}\left(x - \frac{3}{2}\right)$$

Putting $y = 0$ for orthocentre, we get $y = -\frac{9}{10}$.

So, orthocenter is $\left(-\frac{9}{10}, 0\right)$.

5. (3) Equation of tangent to ellipse at M is: $\frac{3x}{2 \times 9} + \frac{\sqrt{6}y}{8} = 1$

Put $y = 0$ as intersection will be on x -axis.

$$\therefore R \equiv (6, 0)$$

$$\text{Equation of normal to parabola at } M \text{ is: } \sqrt{\frac{3}{2}}x + y = 2\sqrt{\frac{3}{2}} + \left(\sqrt{\frac{3}{2}}\right)^3$$

$$\text{Put } y = 0, x = 2 + \frac{3}{2} = \frac{7}{2}$$

$$\therefore Q \equiv \left(\frac{7}{2}, 0\right)$$

$$\therefore \text{Area } (\Delta MQR) = \frac{1}{2} \times \left(6 - \frac{7}{2}\right) \times \sqrt{6} = \frac{5}{4}\sqrt{6} \text{ sq. units.}$$

Area of quadrilateral (MF_1NF_2)

$$= 2 \times \text{Area } (\Delta F_1F_2M)$$

$$= 2 \times \frac{1}{2} \times 2 \times \sqrt{6} = 2\sqrt{6} \text{ sq. units}$$

$$\therefore \text{Required Ratio} = \frac{\frac{5}{4}\sqrt{6}}{2\sqrt{6}} = \frac{5}{8}$$

Matrix Match Type

1. $c \rightarrow r$.

$$x = \sqrt{3} \left(\frac{1-t^2}{1+t^2} \right), y = \frac{2t}{1+t^2}$$

Let $t = \tan \alpha$

$$\therefore \cos 2\alpha = \frac{x}{\sqrt{3}} \text{ and } \sin 2\alpha = y$$

$$\therefore \frac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2 2\alpha = 1, \text{ which is an ellipse.}$$

Note: Solutions of the remaining parts are given in their respective chapters.

Point $(h, 1)$ lies on ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$

$$\Rightarrow \frac{h^2}{6} + \frac{1^2}{3} = 1$$

$$\therefore h = \pm 2$$

Tangent at $(2, 1)$ is $\frac{2x}{6} + \frac{y}{3} = 1$

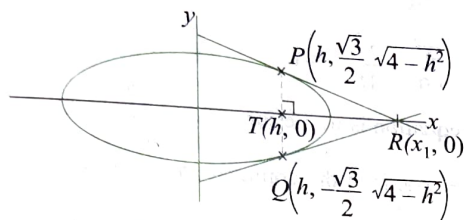
or $x + y = 3$ which is parallel to $x + y = 8$

$$\therefore h = 2$$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (9)



$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\therefore y = \frac{\sqrt{3}}{2} \sqrt{4-h^2} \text{ at } x = h$$

Let $R(x_1, 0)$.

PQ is the chord of contact. So, $\frac{xx_1}{4} = 1$

$$\text{or } x = \frac{4}{x_1}$$

which is the equation of PQ . At $x = h$,

$$\frac{4}{x_1} = h \text{ or } x_1 = \frac{4}{h}$$

$$\Delta(h) = \text{Area of } \triangle PQR = \frac{1}{2} \times PQ \times RT$$

$$= \frac{1}{2} \times \frac{2\sqrt{3}}{2} \sqrt{4-h^2} \times (x_1 - h) = \frac{\sqrt{3}}{2h} (4-h^2)^{3/2}$$

Clearly, $\Delta(h)$ decreases with increase in h .

$$\text{So, } \Delta_1 = \text{Maximum of } \Delta(h) = \frac{45\sqrt{5}}{8} \text{ at } h = \frac{1}{2}$$

$$\text{and } \Delta_2 = \text{Minimum of } \Delta(h) = \frac{9}{2} \text{ at } h = 1$$

$$\text{So, } \frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2 = \frac{8}{\sqrt{5}} \times \frac{45\sqrt{5}}{8} - 8 \times \frac{9}{2} = 45 - 36 = 9$$

$$2. (4) \text{ Ellipse is } \frac{x^2}{9} + \frac{y^2}{5} = 1$$

$$\therefore a^2 e^2 = a^2 - b^2 = 9 - 5 = 4$$

$$\therefore \text{foci are } (2, 0) \text{ and } (-2, 0)$$

$$\therefore f_1 = 2 \text{ and } f_2 = -2$$

Parabola with vertex $O(0, 0)$ and focus $(f_1, 0)$ or $(2, 0)$ is $y^2 = 8x$.

Equation of tangent having slope m_1 is

$$y = m_1 x + \frac{2}{m_1}$$

It passes through $(2f_2, 0)$ or $(-4, 0)$.

$$\therefore 0 = -4m_1 + \frac{2}{m_1} \Rightarrow m_1 = \pm \frac{1}{\sqrt{2}}$$

Parabola with vertex $O(0, 0)$ and focus $(2f_2, 0)$ or $(-4, 0)$ is $y^2 = -16x$.

Equation of tangent having slope m_2 is

$$y = m_2 x - \frac{4}{m_2}$$

It passes through $(f_1, 0)$ or $(2, 0)$.

$$\therefore 0 = 2m_2 - \frac{4}{m_2} \Rightarrow m_2 = \pm \sqrt{2}$$

$$\Rightarrow \frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$$

Chapter 7

Concept Application Exercises

Exercise 7.1

$$1. \sqrt{(x+4)^2 + (y+2)^2} - \sqrt{(x-4)^2 + (y-2)^2} = 8$$

$\underbrace{\hspace{1.5cm}}_{\text{Distance between } P(x,y) \text{ and } A(-4,-2)} - \underbrace{\hspace{1.5cm}}_{\text{Distance between } P(x,y) \text{ and } B(4,2)} = 8$

So, equation represents hyperbola with foci at $A(-4, -2)$ and $B(4, 2)$.

$$2ae = AB = \sqrt{80}$$

Also, $2a = 8$

$$\therefore e = \frac{AB}{8} = \frac{\sqrt{80}}{8} = \frac{\sqrt{5}}{2}$$

2. Taking OA as the x -axis, O as the origin, let the equation of OB be $y = mx$.

Then

$$M \equiv (x_1, 0)$$

The equation of the perpendicular PN is

$$my + x = my_1 + x_1$$

Solving the equations of OB and PN , we get

$$N \equiv \left(\frac{my_1 + x_1}{1 + m^2}, \frac{m(my_1 + x_1)}{1 + m^2} \right)$$

$\therefore$ Area of quadrilateral $OMPN$ (by stair method)

$$= \frac{1}{2} \begin{vmatrix} 0 & 0 \\ x_1 & 0 \\ x_1 & y_1 \\ (my_1 + x_1)/(1 + m^2) & m(my_1 + x_1)/(1 + m^2) \\ 0 & 0 \end{vmatrix}$$

$$= \left[\frac{1}{2} x_1 y_1 + x_1 \frac{m(my_1 + x_1)}{1 + m^2} - y_1 \frac{my_1 + x_1}{1 + m^2} \right]$$

$$= \pm k \text{ (say)}$$

Therefore, the locus of P is

$$mx^2 + 2m^2xy - my^2 \pm 2(1 + m^2)k = 0$$

Here, $h = m^2$

$$a = m$$

$$b = -m$$

$$f = g = 0$$

and $c = \pm 2(1 + m^2)k$

$$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= \pm 2m^2(1 + m^2)k \pm 2m^4(1 + m^2)k^2 \neq 0$$

and $h^2 > ab$

So, the locus is a hyperbola.

$$3. (x-3)^2 + (y+1)^2 = (4x+3y)^2$$

$$\text{or } \sqrt{(x-3)^2 + (y+1)^2} = 5 \left(\frac{|4x+3y|}{\sqrt{4^2 + 3^2}} \right)$$

Clearly, it is hyperbola with focus at $(3, -1)$ and directrix $4x + 3y = 0$.

Transverse axis is perpendicular to directrix and passes through focus.

So, its equation is

$$y + 1 = \frac{3}{4}(x - 3)$$

$$\text{or } 3x - 4y = 13$$

Exercise 7.2

1. We have hyperbola

$$16x^2 - 9y^2 = -144$$

$$\text{or } \frac{x^2}{9} - \frac{y^2}{16} = -1$$

This equation is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Hence, x -axis is the conjugate axis and y -axis is the transverse axis.

Now, $a^2 = 9$, $b^2 = 16$. So, $a = 3$, $b = 4$.

Length of transverse axis $= 2b = 8$

Length of conjugate axis $= 2a = 6$

$$\text{Eccentricity, } e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

Vertices are $(0, \pm b)$ or $(0, \pm 4)$.

Foci are $(0, \pm be)$ or $(0, \pm 5)$.

$$\text{Length of latus rectum} = \frac{2a^2}{b} = \frac{2(3)^2}{4} = \frac{9}{2}$$

Equations of directrices are

$$y = \pm \frac{b}{e}$$

$$\text{or } y = \pm \frac{4}{(5/4)}$$

$$\text{or } y = \pm \frac{16}{5}$$

$$2. \tan 30^\circ = \frac{b^2/a}{ae}$$

$$\text{or } \frac{e}{\sqrt{3}} = e^2 - 1$$

$$\text{or } \sqrt{3}e^2 - e - \sqrt{3} = 0$$

$$\text{or } e = \frac{1 \pm \sqrt{13}}{2\sqrt{3}}$$

$$\text{or } e = \frac{1 + \sqrt{13}}{2\sqrt{3}}$$

3. Distance between the two directrices is

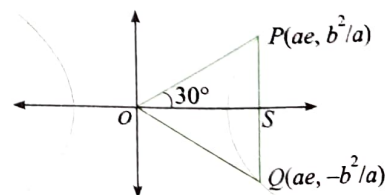
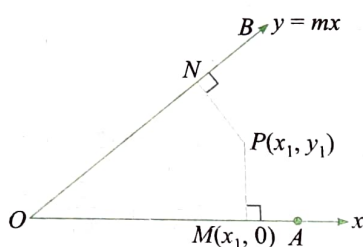
$$\frac{2a}{e} = 10 \text{ units}$$

$$\text{or } a = 5e$$

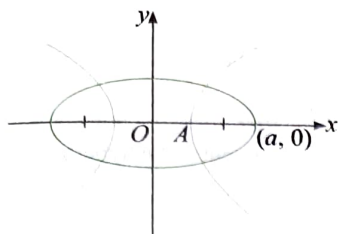
Now, distance between the foci is

$$2ae = 10e^2 = 10(2) = 20$$

(As rectangular hyperbola, $e = \sqrt{2}$)



4. Let the foci of the ellipse be $(\pm ae_1, 0)$ and those of the hyperbola be $(\pm Ae_2, 0)$.



According to the question, we have

$$ae_1 = Ae_2 \quad (1)$$

Also, it is given that the conjugate axis of the hyperbola is equal to the minor axis of the ellipse. Therefore,

$$a^2(1 - e_1^2) = A^2(e_2^2 - 1) \quad (2)$$

From (1) and (2), we have

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 2$$

5. Let any point P on the hyperbola $x^2 - y^2 = a^2$ be (x_1, y_1) .

Now, $SP = ex_1 - a$ and $S'P = ex_1 + a$. Then,

$$\begin{aligned} SP \times S'P &= e^2 x_1^2 - a^2 \\ &= 2x_1^2 - a^2 \\ &= 2x_1^2 - (x_1^2 - y_1^2) \quad [\because \text{Point } (x_1, y_1) \text{ lies on the hyperbola}] \\ &= x_1^2 + y_1^2 \\ &= CP^2 \end{aligned}$$

6. The center of the hyperbola is the midpoint of the line joining the two foci.

So, the coordinates of the center are $((8 + 0)/2, (3 + 3)/2)$, i.e., $(4, 3)$.

Let $2a$ and $2b$ be the lengths of transverse and conjugate axes and let e be the eccentricity.

Then the equation of the hyperbola is

$$\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1 \quad (1)$$

Now, the distance between the foci is $2ae$, i.e.,

$$\sqrt{(8-0)^2 + (3-3)^2} = 2ae \text{ or } ae = 4 \quad (\because e = \frac{4}{3})$$

or $a = 3$

Now, $b^2 = a^2(e^2 - 1) = 9(-1 + \frac{16}{9}) = 7$

Thus, the equation of the hyperbola is

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1$$

7. We have,

$$16(x^2 - 2x) - 3(y^2 - 4y) = 44$$

$$\text{or } 16(x-1)^2 - 3(y-2)^2 = 48$$

$$\text{or } \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$$

Clearly, transverse axis is horizontal, having equation $y - 2 = 0$.

Conjugate axis is $x - 1 = 0$ or $x = 1$.

Centre is $(1, 2)$.

Here, $a = \sqrt{3}$ and $b = 4$.

Thus, length of transverse axis is $2\sqrt{3}$ and that of conjugate axis is 8.

$$\text{Eccentricity, } e = \sqrt{1 + \frac{16}{3}} = \sqrt{\frac{19}{3}}$$

Vertices lie at distance a from centre on transverse axis.

Therefore, vertices are $(1 \pm \sqrt{3}, 2)$.

Foci lie at distance ae from centre on transverse axis.

Therefore, foci are $(1 \pm \sqrt{19}, 2)$.

Equation of directrices are

$$x - 1 = \pm \frac{a}{e}$$

$$\text{or } x - 1 = \pm \frac{3}{\sqrt{19}}$$

8. Squaring and subtracting the given equations, we get

$$x^2 - y^2 = a^2 \text{ which is a rectangular hyperbola.}$$

9. Let the equation of the lines be

$$y = m_1(x - a)$$

$$\text{and } y = m_2(x + a)$$

$$\therefore m_1 m_2 = p$$

$$\therefore y^2 = m_1 m_2 (x^2 - a^2) = p(x^2 - a^2)$$

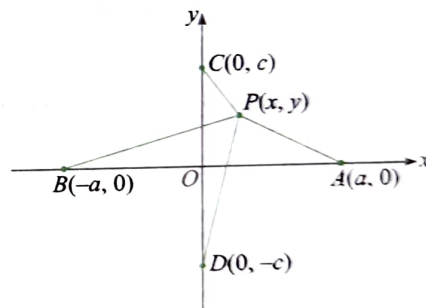
Hence, the locus of the points of intersection is

$$y^2 = p(x^2 - a^2)$$

$$\text{or } px^2 - y^2 = pa^2$$

which is a hyperbola.

10. Taking AOB and COD as the x - and y -axes and their point of intersection O as the origin, clearly, O is the midpoint of AB and CD . Let A be $(a, 0)$ and C be $(0, c)$. Then B is $(-a, 0)$ and D is $(0, -c)$.



Let $P \equiv (x, y)$. Given that

$$PA \cdot PB = PC \cdot PD$$

$$\text{or } \sqrt{(x-a)^2 + y^2} \sqrt{(x+a)^2 + y^2} = \sqrt{x^2 + (y-c)^2} \sqrt{x^2 + (y+c)^2}$$

After simplification, we have

$$x^2 - y^2 = \frac{(a^2 - c^2)}{2}$$

which is a rectangular hyperbola with eccentricity $\sqrt{2}$.

11. The equation of the given hyperbola can be written as

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

Therefore, the equation of the chord of this hyperbola whose midpoint is $(5, 3)$ is

$$\frac{5x}{16} - \frac{3y}{25} - 1 = \frac{5^2}{16} - \frac{9}{25} - 1$$

$$\text{or } 125x - 48y = 481$$

12. Let coordinates of point P on hyperbola be $(a \sec \theta, b \tan \theta)$.

$$\therefore N \equiv (a \sec \theta, 0)$$

Since Q divides AP in the ratio $a^2 : b^2$, we have

$$\therefore Q \equiv \left(\frac{ab^2 + a^3 \sec \theta}{a^2 + b^2}, \frac{a^2 b \tan \theta}{a^2 + b^2} \right).$$

$$\text{Slope of } A'P = \frac{b \tan \theta}{a(\sec \theta + 1)}$$

$$\begin{aligned} \text{Slope of } QN &= \frac{a^2 b \tan \theta}{ab^2 + a^3 \sec \theta - a^3 \sec \theta - ab^2 \sec \theta} \\ &= \frac{a^2 b \tan \theta}{ab^2(1 - \sec \theta)} \end{aligned}$$

$$\text{Slope of } A'P \times \text{Slope of } QN = \frac{a^2 b^2 \tan^2 \theta}{-a^2 b^2 \tan^2 \theta} = -1$$

Hence, QN is perpendicular to $A'P$.

Exercise 7.3

1. We have hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$.

The slope of given line $x - 3y = 4$ is $1/3$.

So, the slope of tangent is -3 .

Therefore, equations of tangents are

$$y = -3x \pm \sqrt{25 \times 9 - 16}$$

$$\text{or } y = -3x \pm \sqrt{209}$$

2. The hyperbola is

$$3x^2 - 2y^2 = 25$$

$$\text{or } \frac{x^2}{25/3} - \frac{y^2}{25/2} = 1 \quad (1)$$

Tangent to the hyperbola having slope m is

$$y = mx + \sqrt{a^2 m^2 - b^2} \quad (2)$$

For the given hyperbola, the tangent having slope m is

$$y = mx + \sqrt{\frac{25}{3} m^2 - \frac{25}{2}}$$

It passes through $(0, 5/2)$. Therefore,

$$\frac{25}{4} = \frac{25}{3} m^2 - \frac{25}{2}$$

$$\text{or } \frac{1}{4} = \frac{2m^2 - 3}{6}$$

$$\text{or } m^2 = \frac{9}{4}$$

$$\text{or } m = \pm \frac{3}{2}$$

Therefore, the equation of tangents are

$$y - \frac{5}{2} = \pm \frac{3}{2}(x - 0)$$

$$\text{or } y = \frac{5}{2} \pm \frac{3}{2}x$$

3. Given equation of hyperbola is

$$9x^2 - 16y^2 = 144$$

$$\text{or } \frac{x^2}{16} - \frac{y^2}{9} = 1$$

The equation of tangent to the hyperbola having slope m is

$$y = mx \pm \sqrt{16m^2 - 9}$$

If it touches the circle, then the distance of this line from the center of the circle is the radius of the circle. Hence,

$$\frac{\sqrt{16m^2 - 9}}{\sqrt{m^2 + 1}} = 3$$

$$\text{or } 9m^2 + 9 = 16m^2 - 9$$

$$\text{or } 7m^2 = 18$$

$$\text{or } m = \pm 3\sqrt{\frac{2}{7}}$$

So, the equation of tangents is

$$y = \pm 3\sqrt{\frac{2}{7}}x \pm \frac{15}{\sqrt{7}}$$

4. The line $y = \alpha x + \beta$ touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{if } \beta^2 = a^2 \alpha^2 - b^2.$$

Hence, the locus of (α, β) is

$$y^2 = a^2 x^2 - b^2$$

$$\text{or } a^2 x^2 - y^2 = b^2$$

$$\text{or } \frac{x^2}{b^2/a^2} - \frac{y^2}{b^2} = 1$$

which is a hyperbola.

5. We have hyperbola

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

Equation of tangent to hyperbola at point $P(3 \sec \theta, 2 \tan \theta)$ is given by

$$\frac{x}{3} \sec \theta - \frac{y}{2} \tan \theta = 1$$

It meets the axes at $A(3 \cos \theta, 0)$ and $B(0, -2 \cot \theta)$.

Let the midpoint of AB be (h, k) . Then

$$2h = 3 \cos \theta \text{ and } 2k = -2 \cot \theta$$

$$\therefore \sec \theta = \frac{3}{2h} \text{ and } \tan \theta = -\frac{1}{k}$$

Squaring and subtracting, we get

$$\frac{9}{4h^2} - \frac{1}{k^2} = 1$$

So, the locus of AB is $\frac{9}{4x^2} - \frac{1}{y^2} = 1$.

6. The equation of the chord of contact of parabola w.r.t. $P(x_1, y_1)$ is given by

$$yy_1 = 2a(x + x_1) \quad (1)$$

Equation (1) touches the curve

$$x^2 - y^2 = c^2 \quad (2)$$

Using the condition of tangency on

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

we get

$$\frac{4a^2 x_1^2}{y_1^2} = c^2 - \frac{4a^2}{y_1^2} \quad \left[\because (1) \text{ is tangent to } \frac{x^2}{c^2} - \frac{y^2}{c^2} = 1 \right]$$

$$\text{or } 4a^2 x_1^2 = 4a^2 c^2 - c^2 y_1^2$$

$$\text{or } 4a^2 x_1^2 + c^2 y_1^2 = 4a^2 c^2$$

Hence, the locus is

$$\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1$$

which is an ellipse.

7. Tangents to parabola intersect the hyperbola at A and B .

Let the point of intersection of tangents at A and B be $P(h, k)$.

So, AB will be chord of contact of hyperbola w.r.t. point P .

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1$$

$$\text{or } \frac{ky}{b^2} = \frac{hx}{a^2} - 1$$

$$\text{or } y = \left(\frac{b^2 h}{ka^2} \right) x - \frac{b^2}{k}$$

This line touches the parabola.

$$\text{So, } -\frac{b^2}{k} = \frac{a}{\frac{b^2 h}{ka^2}} \quad (\text{as } y = mx + c \text{ touches the parabola } y^2 = 4ax \text{ if } c = a/m)$$

$$\Rightarrow -\frac{b^2}{k} = \frac{ka^3}{b^2 h}$$

$$\text{Hence, required locus is } y^2 = -\frac{b^4}{a^3} x.$$

8. Equation of chord of contact with respect to point $P(-4, 2)$ is $\frac{-4x}{a^2} - \frac{2y}{b^2} = 1$ and that with respect to point $(2, 1)$ is

$$\frac{2x}{a^2} - \frac{y}{b^2} = 1.$$

According to given condition,

$$\left(\frac{\frac{4}{a^2}}{\frac{-2}{b^2}} \right) \times \left(\frac{\frac{-2}{a^2}}{\frac{-1}{b^2}} \right) = -1$$

$$\Rightarrow \frac{b^4}{a^4} = \frac{1}{4}$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{1}{2}$$

$$\therefore e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

9. Chord of contact of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ w.r.t. point $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad (1)$$

Eq. (1) can be written as $\frac{x(-x_1)}{a^2} - \frac{y(-y_1)}{b^2} = -1$, which is tangent

to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at point $(-x_1, -y_1)$.

Obviously, points (x_1, y_1) and $(-x_1, -y_1)$ lie on the different

branches of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

10. We have hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Let the coordinates of point R on the hyperbola be $(a \sec \theta, b \tan \theta)$.

$$\text{Equation of tangent at } R \text{ is } \frac{x \sec \theta}{a} - \frac{y \sec \theta}{b} = 1.$$

p = Distance of centre to the tangent

$$= \frac{1}{\sqrt{\frac{\sec^2 \theta}{a^2} + \frac{\tan^2 \theta}{b^2}}} = \frac{ab \cos \theta}{\sqrt{b^2 + a^2 \sin^2 \theta}}$$

Now, $RS = ae \sec \theta - a$ and $RS' = ae \sec \theta + a$.

$$\therefore RS + RS' = 2ae \sec \theta$$

$$\text{So, } (RS + RS')^2 = 4a^2 e^2 \sec^2 \theta$$

$$\begin{aligned} a^2 \left(1 + \frac{b^2}{p^2} \right) &= a^2 \left(1 + \frac{b^2(b^2 + a^2 \sin^2 \theta)}{a^2 b^2 \cos^2 \theta} \right) \\ &= a^2 \left(1 + \frac{b^2}{a^2 \cos^2 \theta} + \tan^2 \theta \right) \\ &= a^2 \sec^2 \theta \left(1 + \frac{b^2}{a^2} \right) \\ &= a^2 e^2 \sec^2 \theta \end{aligned}$$

$$\text{Thus, } (RS + RS')^2 = 4a^2 \left(1 + \frac{b^2}{p^2} \right)$$

Exercise 7.4

1. The angle between the asymptotes is given by

$$\begin{aligned} \tan^{-1} \left(\frac{2ab}{a^2 - b^2} \right) &= \tan^{-1} \left\{ \frac{2(4)(3)}{16 - 9} \right\} \\ &= \tan^{-1} \left(\frac{24}{7} \right) \end{aligned}$$

2. Since the equations of a hyperbola and its asymptotes differ in constant terms only, the pair of asymptotes is given by

$$xy - 3y - 2x + \lambda = 0 \quad (1)$$

where λ is any constant such that it represents two straight lines. Hence,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\text{or } 0 + 2 \times \left(-\frac{3}{2} \right) \times (-1) \times \left(\frac{1}{2} \right) - 0 - 0 - \lambda \left(\frac{1}{2} \right)^2 = 0$$

$$\text{or } \lambda = 6$$

From (1), the asymptotes of the given hyperbola are given by

$$xy - 3y - 2x + 6 = 0$$

$$\text{or } (y - 2)(x - 3) = 0$$

Therefore, the asymptotes are $x - 3 = 0$ and $y - 2 = 0$.

3. Clearly, for the given situation, asymptotes must be perpendicular. So, hyperbola must be rectangular.

Therefore, eccentricity of hyperbola is $\sqrt{2}$

4. Any tangent to the hyperbola forms a triangle with the asymptotes which has constant area ab .

Given

$$ab = a^2 \tan \lambda$$

$$\text{or } \frac{b}{a} = \tan \lambda$$

$$\text{or } e^2 - 1 = \tan^2 \lambda$$

$$\text{or } e^2 = 1 + \tan^2 \lambda = \sec^2 \lambda$$

$$\text{or } e = \sec \lambda$$

5. The foci of the hyperbola lie on $y = x$. So, the major axis is $y = x$.

The major axis of hyperbola bisects the asymptote.

Therefore, the equation of other asymptote is $x = 2y$.

Thus, the equation of hyperbola is $(y - 2x)(x - 2y) + k = 0$.

Given that it passes through (3, 4), we get $k = -10$.

Hence, the required equation is

$$2x^2 + 2y^2 - 5xy + 10 = 0$$

Exercise 7.5

1. Let the perpendicular line cuts the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at point $P(x_1, y_1)$ and the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

at point $Q(x_1, y_2)$.

Normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at point P is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad (1)$$

Normal to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

at Q is

$$\frac{a^2x}{x_1} + \frac{b^2y}{y_2} = a^2 + b^2 \quad (2)$$

In (1) and (2), putting $y = 0$, we get

$$x = \frac{a^2 + b^2}{a^2} x_1$$

Hence, both normals meet on the x -axis.

2. The equation of normal at the point $Q(a \sec \phi, b \tan \phi)$ to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is $ax \cos \phi + by \cot \phi = a^2 + b^2$

The normal (1) meets the x -axis at

$$M\left(\frac{a^2 + b^2}{a} \sec \phi, 0\right)$$

and the y -axis at

$$N\left(0, \frac{a^2 + b^2}{b} \tan \phi\right)$$

Let point P be (h, k) . From the diagram,

$$h = \frac{a^2 + b^2}{a} \sec \phi$$

$$\text{and } k = \frac{a^2 + b^2}{b} \tan \phi$$

Eliminating ϕ by using the relation $\sec^2 \phi - \tan^2 \phi = 1$, we have

$$\left(\frac{ah}{a^2 + b^2}\right)^2 - \left(\frac{bk}{a^2 + b^2}\right)^2 = 1$$

Therefore, $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$ is the required locus of P .

3. Normal at a point $(a \sec \theta, a \tan \theta)$ is

$$x \cos \theta + y \cot \theta = 2a \quad (1)$$

If $P(x_1, y_1)$ is the point of intersection of the tangents at the ends of normal chord (1), then (1) must be the chord of contact of $P(h, k)$ whose equation is given by

$$hx - ky = a^2 \quad (2)$$

Comparing (1) and (2) and eliminating θ , we get

$$\frac{a^2}{4h^2} - \frac{a^2}{4k^2} = 1$$

Hence, the locus is

$$\frac{1}{x^2} - \frac{1}{y^2} = \frac{4}{a^2}$$

4. Equation of normal to $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at point $P(4 \sec \theta, 3 \tan \theta)$ on it is given by

$$\frac{4x}{\sec \theta} + \frac{3y}{\tan \theta} = 16 + 9$$

$$\text{or } 4x \cos \theta + 3y \cot \theta = 25$$

This is of the form $-mx + y = \frac{25\sqrt{3}}{3}$ (given).

Comparing ratios of coefficients, we get

$$\frac{4 \cos \theta}{-m} = \frac{3 \cot \theta}{1} = \sqrt{3}$$

$$\therefore \cot \theta = \frac{1}{\sqrt{3}}$$

$$\therefore m = -\frac{4 \cos \theta}{\sqrt{3}} = -\frac{4}{\sqrt{3}} \cdot \frac{1}{2} = -\frac{2}{\sqrt{3}}$$

5. Normal at point $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$.

It meets the axes at

$$A\left(\frac{(a^2 + b^2)x_1}{a^2}, 0\right) \text{ and } B\left(0, \frac{(a^2 + b^2)y_1}{b^2}\right)$$

$$\text{Area of } \triangle OAB = \frac{1}{2} \left[\frac{(a^2 + b^2)x_1}{a^2} \right] \left[\frac{(a^2 + b^2)y_1}{b^2} \right]$$

$$= \frac{1}{2} \left[\frac{(a^2 + b^2)^2 x_1 y_1}{a^2 b^2} \right]$$

Now, normal is drawn at the extremity of latus rectum.

Hence, $(x_1, y_1) \equiv \left(ae, \frac{b^2}{a}\right)$

$$\therefore \text{Area} = \frac{1}{2} \left[\frac{(a^2 + b^2)^2 b^2 e}{a^2 b^2} \right]$$

$$= \frac{1}{2} \left[\frac{a^4 \left(1 + \frac{b^2}{a^2}\right)^2 e}{a^2} \right]$$

$$= \frac{1}{2} a^2 e^5$$

Exercise 7.6

1. We have hyperbola $(x - 3)(y - 4) = 5$

Clearly, equation of pair of asymptotes is $(x - 3)(y - 4) = 0$.

Thus, asymptotes are $x - 3 = 0$ and $y - 4 = 0$.

Centre of the hyperbola is (3, 4).

Since hyperbola is rectangular, axes are bisectors of asymptotes having slopes ± 1 .

Therefore, equation of transverse axis is $y - 4 = 1(x - 3)$ or $x - y + 1 = 0$ and equation of conjugate axis is $x + y - 7 = 0$.

2. Hyperbola is

$$xy = a^2$$

$$\Rightarrow 2xy - 2a^2 = 0$$

Chord of PQ of hyperbola bisected at point $C(h, k)$ is

$$hy + kx - 2a^2 = 2hk - 2a^2$$

$$\Rightarrow \frac{x}{h} + \frac{y}{k} = 2$$

It meets x -axis at $A(2h, 0)$.

$$\text{Now, } AC = OC = \sqrt{h^2 + k^2}$$

Therefore, triangle OCA is isosceles.

3. We know that the orthocentre of the triangle formed by points $P(t_1)$, $Q(t_2)$ and $R(t_3)$ on hyperbola lies on the same hyperbola and has coordinates $\left(\frac{-c}{t_1 t_2 t_3}, -ct_1 t_2 t_3\right)$.

Also, if points $P(t_1)$, $Q(t_2)$, $R(t_3)$ and $S(t_4)$ are concyclic then $t_1 t_2 t_3 t_4 = 1$.

Therefore, orthocentre of triangle PQR is $\left(-ct_4, \frac{-c}{t_4}\right)$.

4. Equation of hyperbola is $xy = c^2$.

Differentiating w.r.t. x , we get

$$\frac{dy}{dx} = -\frac{y}{x}$$

Thus, slope of normal at any point on the hyperbola is $-\frac{dx}{dy} = \frac{x}{y}$.

If point on the hyperbola is $(x, y) \equiv (ct, c/t)$, then $-\frac{dx}{dy} = t^2$.

So, equation of normal at point $(ct, c/t)$ is

$$y - \frac{c}{t} = t^2(x - ct)$$

$$\Rightarrow ct^4 - xt^3 + yt - c = 0$$

If this normal passes through the point $P(h, k)$ on the plane, then we have

$$ct^4 - ht^3 + kt - c = 0 \quad (1)$$

This is polynomial equation of four degree in variable t .

So, we can get maximum four real roots of the equation.

Hence, maximum four normals can be drawn from point $P(h, k)$.

Now, given that sum of the slopes of normals is λ .

i.e., $t_1^2 + t_2^2 + t_3^2 + t_4^2 = \lambda$; where t_1, t_2, t_3 and t_4 are roots of equation (1).

$$\therefore (t_1 + t_2 + t_3 + t_4)^2 - \sum t_i t_j = 0$$

$$\Rightarrow \left(\frac{h}{c}\right)^2 - 0 = \lambda$$

[From equation (1), using sum of roots and sum of product of roots taking two at a time]

$$\Rightarrow h^2 = c^2 \lambda$$

Therefore, required locus is $x^2 = \lambda c^2$.

2. (1) Let $P \equiv (x, y)$.

$$\therefore \frac{a}{e} + x = \frac{(ex - a) + (ex + a)}{2}$$

$$\therefore \frac{a}{e} + x = ex$$

$$\therefore \frac{16}{5} + x = \frac{5}{4}x$$

$$x = \frac{64}{5}$$

But, point P lies on the left branch.

Therefore, x -coordinate of P is $-\frac{64}{5}$.

3. (3) We have

$$2x^2 + 3y^2 - 8x - 18y + 35 = k$$

$$\text{or } 2(2x^2 - 4x) + 3(y^2 - 6y) + 35 = k$$

$$\text{or } 2(x-2)^2 + 3(y-3)^2 = k$$

For $k = 0$, we get

$$2(x-2)^2 + 3(y-3)^2 = 0$$

which represents the point $(2, 3)$.

4. (2) We have $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$

$$\therefore ax^2 + by^2 + c = 0 \text{ or } x^2 - 5xy + 6y^2 = 0$$

From $x^2 - 5xy + 6y^2 = 0$ we have $(x-2y)(x-3y) = 0$, which represents a pair of straight lines.

If $c = 0$ and a and b have same sign then we have $ax^2 + by^2 = 0$, which is possible. Only if $(x, y) \equiv (0, 0)$.

If $a = b$ and c has sign opposite to a , we have $ax^2 + ay^2 = -c$ or $x^2 + y^2 = -c/a$, which represents the circle.

If a and b have same sign opposite to that of c , then $ax^2 + by^2 = -c$ is equation of ellipse.

5. (2) $e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{\sin^2 \alpha}{\cos^2 \alpha} = \frac{1}{\cos^2 \alpha}$

$$a^2 = \cos^2 \alpha$$

$$\therefore a^2 e^2 = 1$$

Hence, the foci are $(\pm ae, 0) \equiv (\pm 1, 0)$, which are independent of α .

6. (3) If e_1 and e_2 are the eccentricities of two conjugate hyperbolas, then

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

Therefore, $e_1 = \sec \theta$ and $e_2 = \operatorname{cosec} \theta$.

7. (2) Since $e/2$ and $e'/2$ are the eccentricities of a hyperbola and its conjugate hyperbola, we have

$$\frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\text{or } 4 = \frac{e^2 e'^2}{e'^2 + e^2}$$

The line passing through the points $(e, 0)$ and $(0, e')$ is

$$e'x + ey - ee' = 0$$

It is tangent to the circle $x^2 + y^2 = r^2$. Hence,

$$\frac{ee'}{\sqrt{e^2 + e'^2}} = r$$

$$\text{or } r^2 = \frac{e^2 e'^2}{e^2 + e'^2} = 4$$

$$\text{or } r = 2$$

8. (1) The length of transverse axis is

$$2 \sin \theta = 2a$$

$$\text{or } a = \sin \theta$$

Exercises

Single Correct Answer Type

1. (1) Given that

$$\frac{\text{Distance between foci}}{\text{Distance between two directrices}} = \frac{3}{2}$$

$$\text{or } \frac{2ae}{2 \cdot a/e} = \frac{3}{2}$$

$$\text{or } e^2 = \frac{3}{2}$$

$$\text{or } 1 + \frac{b^2}{a^2} = \frac{3}{2}$$

$$\text{or } \frac{b}{a} = \frac{1}{\sqrt{2}}$$

Also, for the ellipse

$$3x^2 + 4y^2 = 12$$

$$\text{or } \frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$a^2 = 4, b^2 = 3$$

$$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$$

Hence, the focus of ellipse is $(2 \times 1/2, 0) \equiv (1, 0)$.

As the hyperbola is confocal with the ellipse, the focus of the hyperbola is $(1, 0)$. Now,

$$ae' = 1$$

$$\text{or } \sin \theta \times e' = 1$$

$$\text{or } e' = \operatorname{cosec} \theta$$

$$\therefore b^2 = a^2(e'^2 - 1) = \sin^2 \theta (\operatorname{cosec}^2 \theta - 1) = \cos^2 \theta$$

Therefore, the equation of hyperbola is

$$\frac{x^2}{\sin^2 \theta} - \frac{y^2}{\cos^2 \theta} = 1$$

$$\text{or } x^2 \operatorname{cosec}^2 \theta - y^2 \sec^2 \theta = 1$$

9. (2) We have

$$ae - \frac{a}{e} = 3 \text{ and } ae + \frac{a}{e} = 5$$

$$\therefore ae = 4 \text{ and } \frac{a}{e} = 1$$

$$\therefore a^2 = 4$$

$$\therefore a = 2$$

$$\therefore e = 2$$

10. (4) $S_1P + S_2P = 2a$ and $S_1P - S_2P = 2A$

$$S_1P = a + A \text{ and } S_2P = (a - A)$$

$$\text{Required quadratic equation is } x^2 - 2ax + (a^2 - A^2) = 0.$$

11. (2) Equation of tangent to ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ having slope m is
 $y = mx + \sqrt{9m^2 + 4}$.

Since it passes through point $(\sqrt{5} \sec \theta, \sqrt{5} \tan \theta)$ on the hyperbola, we have

$$\sqrt{5} \tan \theta = m\sqrt{5} \sec \theta + \sqrt{9m^2 + 4}$$

$$\therefore (\sqrt{5} \tan \theta - m\sqrt{5} \sec \theta)^2 = 9m^2 + 4$$

$$\Rightarrow (9 - 5 \sec^2 \theta)m^2 + 10 \sec \theta \tan \theta m + 4 - 5 \tan^2 \theta = 0$$

This equation has roots m_1 and m_2 , which are slopes of tangents.

$$\text{Now, } m_1 m_2 = \frac{4 - 5 \tan^2 \theta}{9 - 5 \sec^2 \theta} = \frac{4 - 5(\sec^2 \theta - 1)}{9 - 5 \sec^2 \theta} = 1$$

$$\therefore (\tan \alpha)(\tan \beta) = 1$$

$$\Rightarrow \alpha + \beta = \frac{\pi}{2}$$

12. (3) Eccentricity of rectangular hyperbola is $\sqrt{2}$.

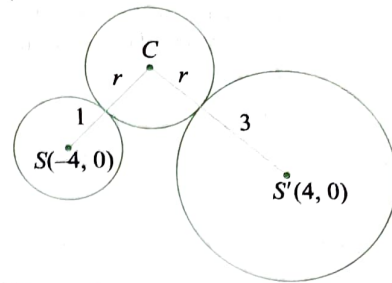
So, equation of hyperbola is

$$\sqrt{(x-1)^2 + (y+1)^2} = \sqrt{2} \frac{|x-y+1|}{\sqrt{1^2 + (-1)^2}}$$

$$\Rightarrow (x-1)^2 + (y+1)^2 = (x-y+1)^2$$

$$\Rightarrow 2xy - 4x + 4y + 1 = 0$$

13. (3)



$$CS = r + 1$$

$$CS' = r + 3$$

$$CS' - CS = 2$$

Locus of C is hyperbola with $(-4, 0)$ and $(4, 0)$ as foci.

Here, $2a = 2$, so $a = 1$.

Also, $2ae = 8$, so $ae = 4$.

$$\therefore b^2 = a^2(e^2 - 1) = 15$$

14. (3) Let the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then, $2a = ae$, i.e., $e = 2$. Therefore,

$$\frac{b^2}{a^2} = e^2 - 1 = 3$$

$$\text{or } \frac{(2b)^2}{(2a)^2} = 3$$

15. (3) We have

$$\frac{2b^2}{a} = 8$$

$$\text{and } 2b = \frac{1}{2}(2ae)$$

$$\therefore \frac{2}{a} \left(\frac{ae}{2} \right)^2 = 8$$

$$\text{or } ae^2 = 16$$

$$\text{Also, } 2 \times \frac{b^2}{a} = 8$$

$$\text{or } b^2 = 4a$$

$$\text{or } a^2(e^2 - 1) = 4a$$

$$\text{or } ae^2 - a = 4$$

From (1) and (2), we have

$$16 - \frac{16}{e^2} = 4$$

$$\text{or } \frac{16}{e^2} = 12$$

$$\text{or } e = \frac{2}{\sqrt{3}}$$

$$16. (4) \tan 30^\circ = \frac{b^2/a}{a + ae}$$

$$\text{or } \frac{1+e}{\sqrt{3}} = e^2 - 1$$

$$\text{or } e - 1 = \frac{1}{\sqrt{3}}$$

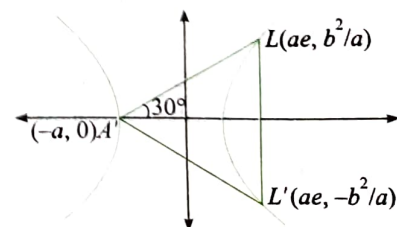
$$\text{or } e = \frac{\sqrt{3} + 1}{\sqrt{3}}$$

17. (1) The given hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{1/3} = 1$$

Its eccentricity e is given by

$$\frac{1}{3} = 1(e^2 - 1)$$



Hence, eccentricity e' of the conjugate hyperbola is given by

$$1 = \frac{1}{3}(e'^2 - 1)$$

or $e'^2 = 4$

or $e' = 2$

18. (2) Eliminating t from the given two equations, we have

$$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

whose eccentricity is

$$e = \sqrt{1 + \frac{48}{16}} = 2$$

19. (2) For the hyperbola

$$\frac{x^2}{5} - \frac{y^2}{5 \cos^2 \alpha} = 1$$

we have

$$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5 \cos^2 \alpha}{5} = 1 + \cos^2 \alpha$$

For the ellipse

$$\frac{x^2}{25 \cos^2 \alpha} + \frac{y^2}{25} = 1$$

we have

$$e_2^2 = 1 - \frac{25 \cos^2 \alpha}{25} = \sin^2 \alpha$$

Given that

$$e_1 = \sqrt{3} e_2$$

$$\therefore e_1^2 = 3e_2^2$$

or $1 + \cos^2 \alpha = 3 \sin^2 \alpha$

or $2 = 4 \sin^2 \alpha$

or $\sin \alpha = \frac{1}{\sqrt{2}}$

20. (2) The equation of the hyperbola is

$$\frac{\{(2x - y + 4)/\sqrt{5}\}^2}{1/2} - \frac{\{(x + 2y - 3)/\sqrt{5}\}^2}{1/3} = 1$$

or $\frac{2}{5}(2x - y + 4)^2 - \frac{3}{5}(x + 2y - 3)^2 = 1$

21. (2) $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$

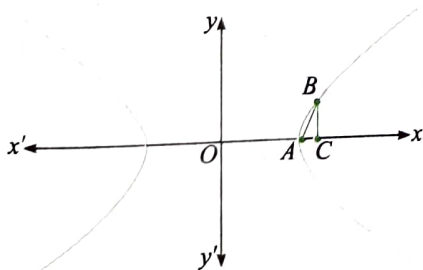
or $(x^2 - 2\sqrt{2}x + 2) - 2(y^2 + 2\sqrt{2}y + 2) = 4$

or $\frac{(x - \sqrt{2})^2}{4} - \frac{(y + \sqrt{2})^2}{2} = 1$

Now B is one of the end points of its latus rectum and C is the focus of the hyperbola nearest to the vertex A .

Clearly, area of $\triangle ABC$ does not change if we consider similar

hyperbola with center at $(0, 0)$ or hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$



Here vertex is $A(2, 0)$.

$$\therefore a^2 e^2 = a^2 + b^2 = 6$$

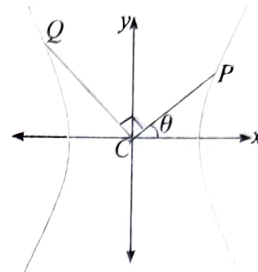
So, one of the foci is $B(\sqrt{6}, 0)$, point C is $(ae, b^2/a)$ or $(\sqrt{6}, 1)$.

$$\therefore AC = \sqrt{6} - 2 \text{ and } BC = 1$$

$$\begin{aligned} \therefore \text{Area of } \triangle ABC &= \frac{1}{2} AC \times BC = \frac{1}{2}(\sqrt{6} - 2) \times 1 \\ &= \frac{\sqrt{3}}{2} - 1 \text{ sq. units.} \end{aligned}$$

22. (4) Let $CP = r_1$ be inclined to the transverse axis at an angle θ so that P is $(r_1 \cos \theta, r_1 \sin \theta)$ and P lies on the hyperbola. It gives

$$r_1^2 \left(\frac{\cos^2 \theta}{a^2} - \frac{\sin^2 \theta}{b^2} \right) = 1$$



Replacing θ by $90^\circ + \theta$, we have

$$r_2^2 \left(\frac{\sin^2 \theta}{a^2} - \frac{\cos^2 \theta}{b^2} \right) = 1$$

or $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{\cos^2 \theta}{a^2} - \frac{\sin^2 \theta}{b^2} + \frac{\sin^2 \theta}{a^2} - \frac{\cos^2 \theta}{b^2}$

or $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \cos^2 \theta \left(\frac{1}{a^2} - \frac{1}{b^2} \right) + \sin^2 \theta \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$

or $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} - \frac{1}{b^2}$

or $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$

23. (3) Homogenizing the hyperbola using the straight line, we get equation of pair of straight lines OP and OQ , which is given by

$$y^2 - x^2 = 4 \left(\frac{\sqrt{3}x + y}{2} \right)^2$$

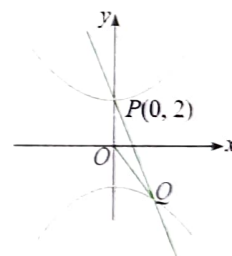
or $y^2 - x^2 = 3x^2 + y^2 + 2\sqrt{3}xy$

or $4x^2 + 2\sqrt{3}xy = 0$

or $x = 0$ and $2x + \sqrt{3}y = 0$

Angle between the lines is

$$\frac{\pi}{2} - \tan^{-1} \left(\frac{2}{\sqrt{3}} \right) = \tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$$



24. (1) Let the variable chord be

$$x \cos \alpha + y \sin \alpha = p \quad (1)$$

Let this chord intersect the hyperbola at A and B . Then the combined equation of OA and OB is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x \cos \alpha + y \sin \alpha}{p} \right)^2$$

$$x^2 \left(\frac{1}{a^2} - \frac{\cos^2 \alpha}{p^2} \right) - y^2 \left(\frac{1}{b^2} + \frac{\sin^2 \alpha}{p^2} \right) - \frac{2 \sin \alpha \cos \alpha}{p} xy = 0$$

This chord subtends a right angle at the center. Therefore, Coefficient of x^2 + Coefficient of y^2 = 0

or $\frac{1}{a^2} - \frac{\cos^2 \alpha}{p^2} - \frac{1}{b^2} - \frac{\sin^2 \alpha}{p^2} = 0$

or $\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2}$

or $p^2 = \frac{a^2 b^2}{b^2 - a^2}$

Hence, p is constant, i.e., it touches the fixed circle.

25. (1) According to the question,

$$\frac{2\sqrt{9m^2 - 49}}{\sqrt{1 + m^2}} = 2$$

$$\begin{aligned} \text{or } 9m^2 - 49 &= 1 + m^2 \\ \text{or } 8m^2 &= 50 \\ \text{or } m &= \pm \frac{5}{2} \end{aligned}$$

26. (1) Any tangent to the hyperbola is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

The given tangent is

$$ax + by = 1$$

Comparing (1) and (2), we have

$$\sec \theta = a^2 \text{ and } \tan \theta = -b^2$$

Eliminating θ , we have

$$a^4 - b^4 = 1$$

$$\text{or } (a^2 - b^2)(a^2 + b^2) = 1$$

$$\text{Also, } a^2 + b^2 = a^2 e^2$$

$$\text{or } a^2 - b^2 = \frac{1}{a^2 e^2}$$

27. (3) $P\left(a \sec \frac{\pi}{6}, b \tan \frac{\pi}{6}\right) \equiv P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$

Therefore, the equation of tangent at P is

$$\frac{x}{\sqrt{3}a/2} - \frac{y}{\sqrt{3}b} = 1$$

$$\therefore \text{Area of triangle} = \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2 \text{ (Given)}$$

$$\text{or } \frac{b}{a} = 4$$

$$\text{or } e^2 = 1 + \frac{b^2}{a^2} = 17$$

28. (3) The equation of the hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

The equation of the tangent is

$$y = mx + \sqrt{9m^2 - 16}$$

$$\text{or } \sqrt{9m^2 - 16} = 2\sqrt{5}$$

$$\text{or } m = \pm 2$$

$$\text{or } a + b = \text{Sum of roots} = 0$$

29. (3) Let the point be (h, k) .

Then the equation of the chord of contact is $hx + ky = 4$.

Since $hx + ky = 4$ is tangent to $xy = 1$,

$$x\left(\frac{4 - hx}{k}\right) = 1$$

has two equal roots.

Therefore, discriminant of $hx^2 - 4x + k = 0$ is 0.

$$\therefore hk = 4$$

Thus, the locus of (h, k) is $xy = 4$.

30. (4) Let the vertex A be $(a \cos \theta, b \sin \theta)$.

Since AC and AB touch the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

BC is the chord of contact. Its equation is

$$\frac{x \cos \theta}{a} - \frac{y \sin \theta}{b} = -1$$

$$\text{or } -\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$

$$\text{or } \frac{x \cos(\pi - \theta)}{a} + \frac{y \sin(\pi - \theta)}{b} = 1$$

which is the equation of the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $(a \cos(\pi - \theta), b \sin(\pi - \theta))$.

Hence, BC touches the given ellipse.

31. (1) Tangent to

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

at $P(3 \sec \theta, 2 \tan \theta)$ is

$$\frac{x}{3} \sec \theta - \frac{y}{2} \tan \theta = 1$$

This is perpendicular to

$$5x + 2y - 10 = 0$$

$$\therefore \frac{2 \sec \theta}{3 \tan \theta} = \frac{2}{5}$$

$$\text{or } \sin \theta = \frac{5}{3}$$

which is not possible.

Hence, there is no such tangent.

32. (2) The equation of tangent at (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

It passes through $(0, -b)$. So,

$$0 + \frac{y_1}{b} = 1 \text{ or } y_1 = b$$

The equation of normal is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 e^2$$

which passes through $(2a\sqrt{2}, 0)$. Hence,

$$\frac{a^2 2a\sqrt{2}}{x_1} = a^2 e^2$$

$$\text{or } x_1 = \frac{2a\sqrt{2}}{e^2}$$

Now, (x_1, y_1) lies on the hyperbola. Therefore,

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\text{or } \frac{8}{e^4} - 1 = 1$$

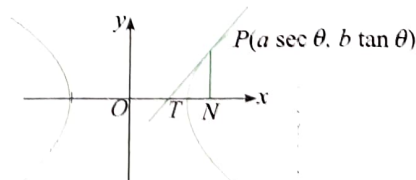
$$\text{or } e^2 = 2$$

$$33. (4) \frac{y^2}{1/16} - \frac{x^2}{1/9} = 1$$

Locus will be the auxiliary circle

$$x^2 + y^2 = \frac{1}{16}$$

34. (2)



Tangent at point P is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

It meets the x -axis at $T(a \cos \theta, 0)$ and the foot of perpendicular from P to the x -axis is $N(a \sec \theta, 0)$.

From the diagram,

$$OT = a \cos \theta \text{ and } ON = a \sec \theta$$

$$\therefore OT \cdot ON = a^2$$

35. (3) Point P is nearest to the given line if the tangent at P is parallel to the given line.

Now, the slope of tangent at $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{18x_1}{24y_1} = \frac{3}{4} \frac{x_1}{y_1} \text{ which must be equal to } -3/2.$$

Therefore,

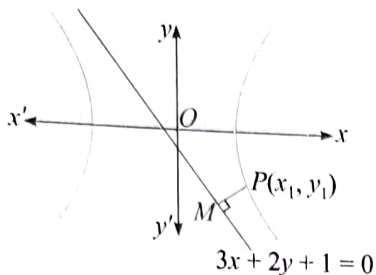
$$\frac{3}{4} \frac{x_1}{y_1} = -\frac{3}{2}$$

$$\text{or } x_1 = -2y_1 \quad (1)$$

Also, (x_1, y_1) lies on the curve. Hence,

$$\frac{x_1^2}{24} - \frac{y_1^2}{18} = 1 \quad (2)$$

Solving (1) and (2), we get two points $(6, -3)$ and $(-6, 3)$ of which $(-6, 3)$ is the nearest.



36. (2) Let the directrix be $x = a/e$ and the focus be $S(ae, 0)$. Let $P(a \sec \theta, b \tan \theta)$ be any point on the curve.

The equation of tangent at P is

$$\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$$

Let F be the intersection point of the tangent of directrix. Then,

$$F = \left(\frac{a}{e}, \frac{b(\sec \theta - e)}{e \tan \theta} \right)$$

$$\therefore m_{SF} = \frac{b(\sec \theta - e)}{-a \tan \theta (e^2 - 1)}$$

$$\text{and } m_{PS} = \frac{b \tan \theta}{a(\sec \theta - e)}$$

$$\therefore m_{SF} \cdot m_{PS} = -1$$

37. (3) Let $y = mx \pm \sqrt{m^2 a^2 - a^2}$ be two tangents that pass through (h, k) . Then

$$(k - mh)^2 = m^2 a^2 - a^2$$

$$\text{or } m^2 (h^2 - a^2) - 2khm + k^2 + a^2 = 0$$

$$\text{or } m_1 + m_2 = \frac{2kh}{h^2 - a^2}$$

$$\text{and } m_1 m_2 = \frac{k^2 + a^2}{h^2 - a^2}$$

$$\text{Now, } \tan 45^\circ = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\text{or } 1 = \frac{(m_1 + m_2)^2 - 4m_1 m_2}{(1 + m_1 m_2)^2}$$

$$\text{or } \left(1 + \frac{k^2 + a^2}{h^2 - a^2} \right)^2 = \left(\frac{2kh}{h^2 - a^2} \right)^2 - 4 \left(\frac{k^2 + a^2}{h^2 - a^2} \right)$$

$$\text{or } (h^2 + k^2)^2 = 4h^2 k^2 - 4(k^2 + a^2)(h^2 - a^2)$$

$$\text{or } (x^2 + y^2)^2 = 4(a^2 y^2 - a^2 x^2 + a^4)$$

$$\text{or } (x^2 + y^2)^2 + 4a^2 (x^2 - y^2) = 4a^4$$

38. (3) The fourth vertex of the parallelogram lies on the circumcircle.

Therefore, the parallelogram is cyclic,

i.e., the parallelogram is a rectangle,

i.e., the tangents are perpendicular.

Therefore, the locus of P is the director circle.

39. (1) Director circle of the circle $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$.

The semi-transverse axis is $\sqrt{3}a$.

The radius of the circle is $\sqrt{2}a$.

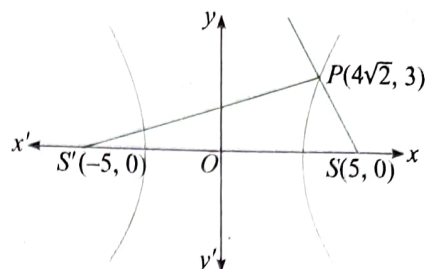
Hence, director circle and hyperbola do not intersect.

40. (4) The given hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$\therefore e = \frac{5}{4}$$

Its foci are $(\pm 5, 0)$.



The ray is incident at $P(4\sqrt{2}, 3)$.

The incident ray passes through $(5, 0)$; so, the reflected ray must pass through $(-5, 0)$.

Its equation is

$$\frac{y - 0}{x + 5} = \frac{3}{4\sqrt{2} + 5}$$

$$\text{or } 3x - y(4\sqrt{2} + 5) + 15 = 0$$

41. (3) Let $P(h, k)$ be any point. The chord of contact of P w.r.t. the hyperbola is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \quad (1)$$

The chord of contact of P w.r.t. the auxiliary circle is

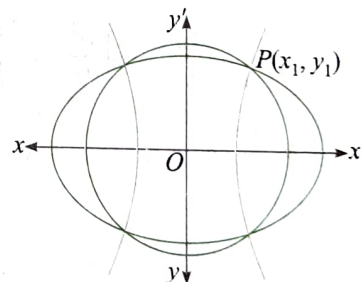
$$hx + ky = a^2 \quad (2)$$

$$\text{Now, } \frac{h}{a^2} \times \frac{b^2}{k} \times \left(-\frac{h}{k} \right) = -1$$

$$\text{or } \frac{h^2}{a^2} - \frac{k^2}{b^2} = 0$$

Therefore, P lies on one of the asymptotes.

42. (1)



Since ellipse and hyperbola intersect orthogonally, they are confocal.

Hence, $a = 2$ (equating foci).

Let the point of intersection in the first quadrant be $P(x_1, y_1)$.

P lies on both the curves. Therefore,

$$4x_1^2 + 9y_1^2 = 36 \text{ and } 4x_1^2 - y_1^2 = 4$$

Adding these two results, we get

$$8(x_1^2 + y_1^2) = 40$$

$$\text{or } x_1^2 + y_1^2 = 5$$

$$\text{or } r = \sqrt{5}$$

Hence, the equation of the circle is

$$x^2 + y^2 = 5$$

43. (2) The chord of contact of tangents from (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

It meets the axes at the points $(a^2/x_1, 0)$ and $(0, b^2/y_1)$.

$$\text{Area of triangle} = \frac{1}{2} \frac{a^2 b^2}{x_1 y_1} = k \text{ (constant)}$$

$$\text{or } x_1 y_1 = \frac{a^2 b^2}{2k} = c^2 \text{ (c is constant)}$$

Therefore, $xy = c^2$ is the required locus.

44. (2) Let a pair of tangents be drawn from the point (x_1, y_1) to the hyperbola

$$x^2 - y^2 = 9$$

Then the chord of contact will be

$$xx_1 - yy_1 = 9 \quad (1)$$

But the given chord of contact is

$$x = 9 \quad (2)$$

As (1) and (2) represent the same line, these equations should be identical and, hence,

$$\frac{x_1}{1} = -\frac{y_1}{0} = \frac{9}{9} \text{ or } x_1 = 1, y_1 = 0$$

Therefore, the equation of pair of tangents drawn from $(1, 0)$ to $x^2 - y^2 = 9$ is

$$(x^2 - y^2 - 9)(1^2 - 0^2 - 9) = (x \cdot 1 - y \cdot 0 - 9)^2 \quad (\text{Using } SS_1 = T^2)$$

$$\text{or } (x^2 - y^2 - 9)(-8) = (x - 9)^2$$

$$\text{or } -8x^2 + 8y^2 + 72 = x^2 - 18x + 81$$

$$\text{or } 9x^2 - 8y^2 - 18x + 9 = 0$$

45. (2) Equation of tangent at P is

$$\frac{xh}{a^2} - \frac{yk}{b^2} = 1$$

$$\therefore x = \left(1 + \frac{yk}{b^2}\right) \frac{a^2}{h}$$

Putting into the equation $x^2 + y^2 = a^2$, we get

$$\left(1 + \frac{y^2 k^2}{b^4} + \frac{2yk}{b^2}\right) \frac{a^4}{h^2} + y^2 = a^2$$

$$\Rightarrow y^2 \left(\frac{k^2 a^4}{b^4 h^2} + 1 \right) + \frac{2ka^4}{b^2 h^2} y + \left(\frac{a^4}{h^2} - a^2 \right) = 0$$

$$\Rightarrow \frac{1}{y_1} + \frac{1}{y_2} = \frac{y_1 + y_2}{y_1 y_2}$$

$$= \frac{-2ka^4}{\frac{b^2 h^2}{a^4 - a^2 h^2}} = -\frac{2ka^2}{b^2 (a^2 - h^2)}$$

$$= -\frac{2k}{b^2 \left(1 - \frac{h^2}{a^2}\right)} = -\frac{2k}{b^2 \left(-\frac{k^2}{b^2}\right)} = \frac{2}{k}$$

46. (4) Normals at $P(\theta)$ and $Q(\phi)$ are

$$ax \cos \theta + by \cot \theta = a^2 + b^2$$

$$ax \cos \phi + by \cot \phi = a^2 + b^2$$

$$\text{where } \phi = \frac{\pi}{2} - \theta$$

These normals pass through (h, k) . Therefore,

$$ah \cos \theta + bk \cot \theta = a^2 + b^2$$

$$\text{and } ah \sin \theta + bk \tan \theta = a^2 + b^2$$

Eliminating h , we have

$$bk(\cot \theta \sin \theta - \tan \theta \cos \theta) = (a^2 + b^2)(\sin \theta - \cos \theta)$$

$$\text{or } k = -\left(\frac{a^2 + b^2}{b}\right)$$

47. (4) The equation of the normal to the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{1} = 1$$

at $(2 \sec \theta, \tan \theta)$ is $2x \cos \theta + y \cot \theta = 5$.

The slope of the normal is

$$-2 \sin \theta = -1$$

$$\text{or } \sin \theta = \frac{1}{2} \text{ or } \theta = \frac{\pi}{6}$$

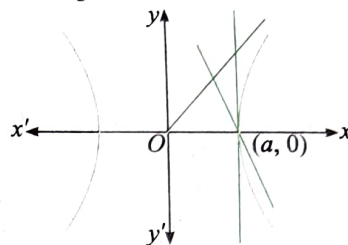
$$Y\text{-intercept of the normal} = \frac{5}{\cot \theta} = \frac{5}{\sqrt{3}}$$

As it touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\text{we have } a^2 + b^2 = \frac{25}{3}$$

48. (2)



The line $y + \lambda(x - a) = 0$ will intersect the portion of the asymptote in the first quadrant only if its slope is negative. Hence,

$$-\lambda < 0$$

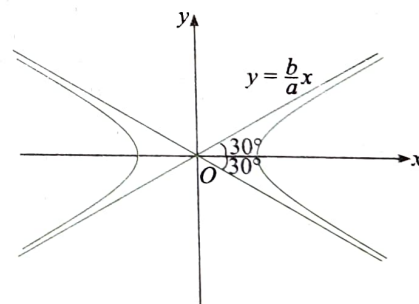
$$\text{or } \lambda > 0$$

$$\therefore \lambda \in (0, \infty)$$

49. (4) The product of perpendiculars drawn from the foci upon any of its tangents is 9. Therefore,

$$b^2 = 9$$

$$\text{Also, } \frac{b}{a} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$



$$\therefore a^2 = 3b^2 = 27$$

Therefore, the required locus is the director circle of the hyperbola which is given by

$$x^2 + y^2 = 27 - 9$$

$$\text{or } x^2 + y^2 = 18$$

If $b/a = \tan 60^\circ$, then

$$a^2 = \frac{b^2}{3} = \frac{9}{3} = 3$$

Hence, the required locus is $x^2 + y^2 = 3 - 9 = -6$ which is not possible.

50. (2) $NP = \frac{4}{5}\sqrt{x_1^2 - 25}$

Point Q is on

$$y = \frac{4}{5}x$$

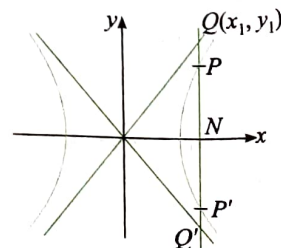
$$NQ = \frac{4}{5}x_1$$

$$PQ = NQ - NP$$

$$= \frac{4}{5}(x_1 - \sqrt{x_1^2 - 25})$$

$$P'Q = \frac{4}{5}(x_1 + \sqrt{x_1^2 - 25})$$

$$PQ \cdot P'Q = 16$$



51. (2) Let the asymptotes be $2x + 3y + \lambda_1 = 0$ and $x + 2y + \lambda_2 = 0$.

It will pass through the center (1, 2). Hence,

$$\lambda_1 = -8, \lambda_2 = -5$$

The equation of the hyperbola is

$$(2x + 3y - 8)(x + 2y - 5) + \lambda = 0$$

It passes through (2, 4). Therefore,

$$(4 + 12 - 8)(2 + 8 - 5) + \lambda = 0 \text{ or } \lambda = -40$$

Hence, equation of hyperbola is

$$(2x + 3y - 8)(x + 2y - 5) = 40$$

52. (3) The slopes of asymptotes are

$$m_1 = \frac{b_1}{a_1}, m_2 = \frac{b_2}{a_2}$$

According to the question,

$$m_1 m_2 = -1$$

$$\text{or } a_1 a_2 + b_1 b_2 = 0$$

53. (3) For the equation $S + K = 0$ to represent a pair of lines,

$$\begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & 1+K \end{vmatrix} = 0$$

$$\text{or } K = -22$$

54. (1) For two distinct tangents on different branches, the point should lie on the line $y = 2$ and between A and B (where A and B are the points on the asymptotes).

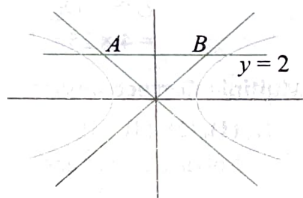
The equations of asymptotes are

$$4x = \pm 3y.$$

Solving with $y = 2$, we have

$$x = \pm \frac{3}{2}$$

$$\therefore -\frac{3}{2} < \alpha < \frac{3}{2}$$



55. (4) Transverse axis is the equation of the angle bisector containing point (2, 3), which is given by

$$\frac{3x - 4y + 5}{5} = \frac{12x + 5y - 40}{13}$$

$$\text{or } 21x + 77y = 265$$

56. (4) Let $P(x_1, y_1)$ be a point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The chord of contact of tangents from P to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2 \text{ is given by}$$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 2 \quad (1)$$

The equations of the asymptotes are

$$\frac{x}{a} - \frac{y}{b} = 0$$

$$\text{and } \frac{x}{a} + \frac{y}{b} = 0$$

The points of intersection of (1) with the two asymptotes are given by

$$x_1 = \frac{2a}{(x_1/a) - (y_1/b)}, y_1 = \frac{2b}{(x_1/a) - (y_1/b)}$$

$$x_2 = \frac{2a}{(x_1/a) + (y_1/b)}, y_2 = \frac{-2b}{(x_1/a) + (y_1/b)}$$

$$\text{Area of the said triangle} = \frac{1}{2} |x_1 y_2 - x_2 y_1|$$

$$= \frac{1}{2} \left| \frac{4ab \times 2}{(x_1^2/a^2) - (y_1^2/b^2)} \right| = 4ab$$

57. (1) Let the equation of the asymptotes be

$$2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0 \quad (1)$$

This equation represents a pair of straight lines. Therefore,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

We have

$$4\lambda + 25 - \frac{25}{2} - 8 - \lambda \frac{25}{4} = 0$$

$$\text{or } -\frac{9\lambda}{4} + \frac{9}{2} = 0$$

$$\text{or } \lambda = 2$$

Putting the value of λ in (1), we get

$$2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$$

This is the equation of the asymptotes.

58. (1) The given hyperbola is

$$xy - hx - ky = 0 \quad (1)$$

The equation of asymptotes is given by

$$xy - hx - ky + c = 0 \quad (2)$$

Equation (2) gives a pair of straight lines. So,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$$

$$\text{or } \begin{vmatrix} 0 & 1/2 & -h/2 \\ 1/2 & 0 & -k/2 \\ -h/2 & -k/2 & c \end{vmatrix} = 0$$

$$\text{or } \frac{hk}{8} + \frac{hk}{8} - \frac{c}{4} = 0$$

$$\text{or } c = hk$$

Hence, the asymptotes are

$$xy - hx - ky + hk = 0$$

$$\text{or } (x - k)(y - h) = 0$$

59. (4) The asymptotes of a rectangular hyperbola are perpendicular to each other.

Given one asymptote

$$x + y + c = 0$$

Let the other asymptote be

$$x - y + \lambda = 0$$

We also know that the asymptotes pass through the center of the hyperbola. Therefore, the line $2x - y = 0$ and the asymptotes must be concurrent.

Thus, we have

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 1 & c \\ 1 & -1 & \lambda \end{vmatrix} = 0$$

$$\text{or } \lambda = -\frac{c}{3}$$

60. (4) The equation of the rectangular hyperbola is

$$(x - 3)(y - 5) + \lambda = 0$$

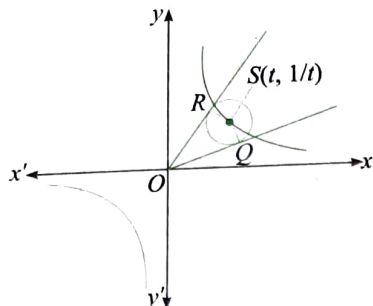
It passes through (7, 8). Hence,

$$4 \times 3 + \lambda \text{ or } \lambda = -12$$

$$\therefore xy - 5x - 3y + 15 - 12 = 0$$

$$\text{or } xy - 3y - 5x + 3 = 0$$

61. (2) Let S be a point on the rectangular hyperbola [say $(t, 1/t)$].
Now, the circumcircle of ΔOQR also passes through S .



Therefore, the circumcenter is the midpoint of OS . Hence,

$$x = \frac{t}{2}, y = \frac{1}{2t}$$

So, the locus of the circumcenter is $xy = 1/4$.

62. (1) The points are such that one of the points is the orthocenter of the triangle formed by other three points. When the vertices of a triangle lie on a rectangular hyperbola, the orthocenter also lies on the same hyperbola.

63. (2) Required area = $4 \times \text{Area of } \Delta S_2OS_4 = 4 \times \frac{1}{2} ae \times be_1$

$$= 4 \times \frac{1}{2} \times 2 \times 3 \times ee_1$$

$$b^2 = a^2(e^2 - 1)$$

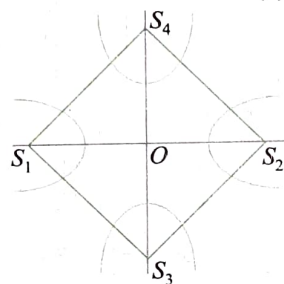
$$\text{or } e^2 = \frac{9}{4} + 1 = \frac{13}{4}$$

$$\text{Also, } \frac{1}{e_1^2} = 1 - \frac{1}{e^2} = 1 - \frac{4}{13} = \frac{9}{13}$$

$$\text{or } e_1^2 = \frac{13}{9}$$

$$\text{Required area} = 12 \times \frac{\sqrt{13}}{2} \times \frac{\sqrt{13}}{3}$$

$$= 2 \times 13 = 26$$



64. (1) For $\lambda = -3$, the equation becomes

$$x^2 + y^2 - 3xy = 0$$

which represents a pair of lines through the origin.

65. (1) The midpoint of the chord is $((x_1 + x_2)/2, (y_1 + y_2)/2)$.

The equation of the chord in terms of its midpoint is

$$T = S_1$$

$$\text{or } x \left(\frac{y_1 + y_2}{2} \right) + y \left(\frac{x_1 + x_2}{2} \right) - 2c^2 = \left(\frac{x_1 + x_2}{2} \right) \left(\frac{y_1 + y_2}{2} \right) - 2c^2$$

$$\text{or } x(y_1 + y_2) + y(x_1 + x_2) = (x_1 + x_2)(y_1 + y_2)$$

$$\text{or } \frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$$

66. (4) Let the foot of perpendicular from $O(0, 0)$ to the tangent to the hyperbola be $P(h, k)$.

$$\text{Slope of } OP = \frac{k}{h}$$

Then the equation of tangent to the hyperbola is

$$y - k = -\frac{h}{k}(x - h)$$

$$\text{or } hx + ky = h^2 + k^2$$

Solving it with $xy = 1$, we have

$$hx + \frac{k}{x} = h^2 + k^2$$

$$\text{or } hx^2 - (h^2 + k^2)x + k = 0$$

This equation must have real and equal roots. Hence,

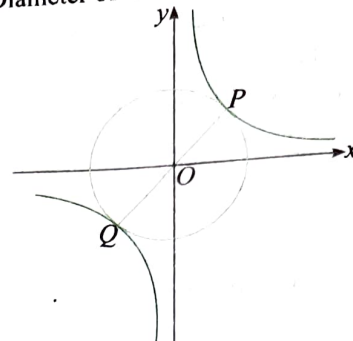
$$D = 0$$

$$\text{or } (h^2 + k^2)^2 - 4hk = 0$$

$$\text{or } (x^2 + y^2)^2 = 4xy$$

67. (2) From the diagram,

$$PQ = \text{Diameter of circle} = 2$$



68. (2) $x - 2 = m$

$$y + 1 = \frac{4}{m}$$

$$\therefore (x - 2)(y + 1) = 4$$

$$\text{or } XY = 4$$

$$\text{where } X = x - 2, Y = y + 1$$

$$S \equiv (x - 2)^2 + (y + 1)^2$$

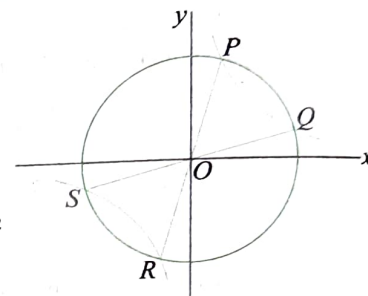
$$= 25$$

$$\text{or } X^2 + Y^2 = 25$$

Curve C and circle S both are concentric. Therefore,

$$OP^2 + OQ^2 + OR^2 + OS^2 = 4r^2$$

$$= 4 \times 25 = 100$$



Multiple Correct Answers Type

1. (1), (2), (3), (4)

Solving $xy = c^2$ and $x^2 + y^2 = a^2$, we have

$$x^2 + \frac{c^4}{x^2} = a^2$$

$$\text{or } x^4 - a^2x^2 + c^4 = 0$$

$$\therefore \Sigma x_i = 0 \text{ and } x_1x_2x_3x_4 = c^4$$

Similarly, if we eliminate y , then $\Sigma y_i = 0$ and $y_1y_2y_3y_4 = c^4$.

2. (2), (3), (4)

$$(x - \alpha)^2 + (y - \beta)^2 = k(lx + my + n)^2$$

$$\text{or } \sqrt{(x - \alpha)^2 + (y - \beta)^2} = \sqrt{k} \sqrt{l^2 + m^2} \frac{(lx + my + n)}{\sqrt{l^2 + m^2}}$$

$$\text{or } \frac{PS}{PM} = \sqrt{k} \sqrt{l^2 + m^2}$$

where $P(x, y)$ is any point on the curve.

Fixed $S(\alpha, \beta)$ is focus and fixed line $lx + my + n = 0$ is directrix.

If $k(l^2 + m^2) = 1$, P lies on a parabola.

If $k(l^2 + m^2) < 1$, P lies on an ellipse.

If $k(l^2 + m^2) > 1$, P lies on a hyperbola.

If $k = 0$, P lies on a point circle.

3. (1), (3)

We have foci $S_1(5, 12)$ and $S_2(24, 7)$.

Hyperbola passes through $P(0, 0)$.

$$\therefore |PS_1 - PS_2| = 2a$$

$$2a = K$$

$$\text{or } 2a = \left| \sqrt{(24 - 0)^2 + (7 - 0)^2} - \sqrt{12^2 + 5^2} \right| = 12$$

$$\therefore a = 6$$

$$S_1S_2 = 2ae = \sqrt{(24 - 5)^2 + (12 - 7)^2} = \sqrt{386}$$

$$e = \frac{\sqrt{386}}{12}$$

$$LR = \frac{2b^2}{a} = \frac{2a^2(e^2 - 1)}{a}$$

$$= 2 \times 6 \left(\frac{386}{144} - 1 \right) = \frac{121}{6}$$

4. (1), (2), (3), (4)

The given hyperbola can be written as

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1$$

or $\frac{X^2}{16} - \frac{Y^2}{9} = 1$ (where $X = x - 1$, $Y = y - 1$)

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

Directrices are $X = \pm a/e$. Therefore,

$$x - 1 = \pm \frac{16}{5}$$

or $x = \frac{21}{5}$ and $x = -\frac{11}{5}$

Length of latus rectum = $\frac{2b^2}{a} = \frac{9}{2}$

The foci are given by

$$X = \pm ae, Y = 0$$

or $(6, 1), (-4, 1)$

5. (1), (3)

For the given ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

we have

$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

Hence, the eccentricity of the hyperbola is $5/3$.

Let the hyperbola be

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$

Then $B^2 = A^2(e^2 - 1) = A^2\left(\frac{25}{9} - 1\right) = \frac{16}{9}A^2$

Therefore, the equation of the hyperbola is

$$\frac{x^2}{A^2} - \frac{9y^2}{16A^2} = 1$$

As it passes through $(3, 0)$, we get $A^2 = 9$ and $B^2 = 16$.

The equation is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

The foci of the hyperbola are $(\pm ae, 0) \equiv (\pm 5, 0)$.

The vertex of the hyperbola is $(3, 0)$.

6. (1), (2), (4)

For the ellipse,

$$a = 5 \text{ and } e = \sqrt{\frac{25-9}{25}} = \frac{4}{5}$$

$$\therefore ae = 4$$

Hence, the foci are $(-4, 0)$ and $(4, 0)$.

For the hyperbola,

$$ae = 4, e = 2$$

$$\therefore a = 2$$

$$b^2 = 4(4 - 1) = 12$$

or $b = \sqrt{12}$

7. (2), (4)

$$\frac{dx}{dy} = \frac{3y}{2x}$$

or $\int 2x dx = \int 3y dy$

or $x^2 = \frac{3y^2}{2} + c$

or $\frac{x^2}{3} - \frac{y^2}{2} = \frac{c}{3}$

If c is positive, then

$$e = \sqrt{1 + \frac{2}{3}} = \sqrt{\frac{5}{3}}$$

If c is negative, then

$$e = \sqrt{1 + \frac{3}{2}} = \sqrt{\frac{5}{2}}$$

8. (1), (2)

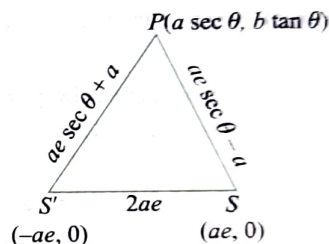
Let (h, k) be the excenter. Then,

$$h = \frac{ae(ae \sec \theta + a) - ae(ae \sec \theta - a) - 2ae(a \sec \theta)}{2ae(\sec \theta - 1)} = -a$$

or $x = -a$ (for $S'P > SP$)

Similarly, $x = a$ (for $S'P < SP$).

Therefore, the locus is $x^2 = a^2$.



Again, let (h, k) be the excenter opposite $\angle S$. Then,

$$h = \frac{2a^2e \sec \theta + a^2e^2 \sec \theta + a^2e + a^2e^2 \sec \theta - a^2e}{2a + 2ae}$$

$$= ae \sec \theta$$

and $k = \frac{2aeb \tan \theta}{2a + 2ae}$

Therefore, the locus is a hyperbola.

9. (1), (4)

End points of latus rectum of ellipse are $(\pm\sqrt{2}, \pm 1)$.

Let hyperbola be $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$.

End points of latus rectum of hyperbola coincide with those of ellipse.

$$\therefore Ae = \sqrt{2} \text{ and } \frac{B^2}{A} = 1$$

Also, $B^2 = A^2(e^2 - 1)$

$$\Rightarrow A = A^2e^2 - A^2$$

Eliminating B , we get

$$\Rightarrow A^2 + A - 2 = 0$$

$$\therefore A = 1 \text{ and } e = \sqrt{2}$$

Let hyperbola be $\frac{y^2}{B^2} - \frac{x^2}{A^2} = 1$.

$$\therefore Be = 1, \frac{A^2}{B} = \sqrt{2}$$

$$\text{Also, } e = \sqrt{1 + \frac{A^2}{B^2}}$$

$$\therefore \frac{1}{B^2} = 1 + \frac{\sqrt{2}}{B}$$

$$\therefore B^2 + \sqrt{2}B - 1 = 0$$

$$\therefore B = \frac{-\sqrt{2} + \sqrt{6}}{2}$$

$$\therefore e = \frac{2}{\sqrt{2}(\sqrt{3}-1)} = \frac{\sqrt{2}(\sqrt{3}+1)}{2} = \frac{\sqrt{3}+1}{\sqrt{2}}$$

10. (2)

The locus of the point of intersection of perpendicular tangents is director circle for

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equation of director circle is $x^2 + y^2 = a^2 - b^2$ which is real if $a > b$.

11. (2), (4)

Differentiating $xy = 1$ w.r.t. x , we have

$$\frac{dy}{dx} = -\frac{1}{x^2} < 0$$

Hence, the slope of normal at any point $P(x_1, y_1)$ is

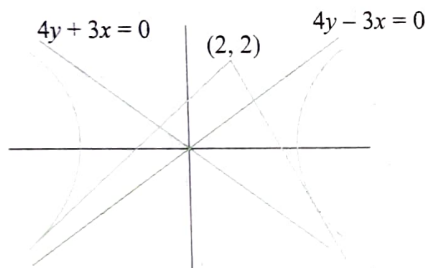
$$x_1^2 > 0.$$

Therefore, the slope of normal must always be positive.

Hence, the possible lines are (2) and (4).

12. (3), (4)

The equations of asymptotes are $4y - 3x = 0$ and $4y + 3x = 0$.



As point (2, 2) lies above the asymptote, the points of contact of the tangents are in the third and fourth quadrants.

13. (1), (2), (3)

The locus of the point of intersection of perpendicular tangents is director circle $x^2 + y^2 = a^2 - b^2$. Now,

$$e^2 = 1 + \frac{b^2}{a^2}$$

If $a^2 > b^2$, then there are infinite (or more than 1) points on the circle, i.e., $e^2 < 2$ or $e < \sqrt{2}$.

If $a^2 < b^2$, there does not exist any point on the plane, i.e., $e^2 > 2$ or $e > \sqrt{2}$.

If $a^2 = b^2$, there is exactly one point (center of the hyperbola), i.e., $e = \sqrt{2}$.

14. (1), (2), (3), (4)

Normal at point $P(2 \sec \theta, 2 \tan \theta)$ is

$$\frac{2x}{\sec \theta} + \frac{2y}{\tan \theta} = 8$$

It meets the axes at points $G(4 \sec \theta, 0)$ and $g(0, 4 \tan \theta)$. Then,

$$PG = \sqrt{4 \sec^2 \theta + 4 \tan^2 \theta}$$

$$Pg = \sqrt{4 \sec^2 \theta + 4 \tan^2 \theta}$$

$$PC = \sqrt{4 \sec^2 \theta + 4 \tan^2 \theta}$$

$$Gg = \sqrt{16 \sec^2 \theta + 16 \tan^2 \theta} = 2\sqrt{4 \sec^2 \theta + 4 \tan^2 \theta} = 2PC$$

15. (1), (2)

$$y = \frac{x+9}{x+5} = 1 + \frac{4}{x+5}$$

$$\therefore \frac{dy}{dx} = \frac{-4}{(x+5)^2}$$

$$\text{At } (x_1, y_1), \frac{dy}{dx} = \frac{-4}{(x_1+5)^2}$$

Equation of tangent at (x_1, y_1) ,

$$y - y_1 = \frac{-4}{(x_1+5)^2} (x - x_1)$$

$$\Rightarrow y - 1 - \frac{4}{x_1+5} = \frac{-4}{(x_1+5)^2} (x - x_1)$$

Since it passes through (0, 0),

$$(x_1+5)^2 + 4(x_1+5) + 4x_1 = 0$$

$$\Rightarrow x_1^2 + 18x_1 + 45 = 0$$

$$\therefore x_1 = -15 \text{ or } x_1 = -3$$

So, equations of required tangents are $x + 25y = 0$ or $x + y = 0$.

16. (2), (4)

Equation of tangent parallel to the line $2x + y + 8 = 0$ is $y = -2x + c$.

Solving with hyperbola $x^2 - y^2 = 3$, we get

$$x^2 - (-2x + c)^2 = 3$$

Discriminant of this quadratic is zero.

$$\therefore c = \pm 3$$

So, equations of tangents are

$$2x + y = \pm 3 \quad (1)$$

Let (x_1, y_1) be the point of contact, then equation of tangent is

$$xx_1 - yy_1 = 3 \quad (2)$$

Comparing Eqs. (1) and (2), we get

$$\frac{2}{x_1} = \frac{1}{-y_1} = \pm \frac{3}{3}$$

$$\therefore x_1 = \pm 2 \text{ and } y_1 = \mp 1$$

So, points of contact are (2, -1) and (-2, 1).

17. (1), (2), (4)

Given hyperbola is

$$x^2 - 9y^2 = 9$$

$$\text{or } \frac{x^2}{9} - \frac{y^2}{1} = 1$$

Equation of tangent is

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

This tangent passes through (3, 2).

$$\text{So, } 2 = 3m \pm \sqrt{9m^2 - 1}$$

$$\Rightarrow 4 + 9m^2 - 12m = 9m^2 - 1$$

$$\therefore m_1 = 5/12 \text{ and } m_2 = \infty$$

So, equations of tangents are

$$y - 2 = (5/12)(x - 3)$$

$$\Rightarrow 5x - 12y + 9 = 0 \quad (1)$$

$$\text{and } x = 3 \quad (2)$$

Now, equation of chord of contact w.r.t. point $P(3, 2)$ is

$$T = 0$$

$$\Rightarrow 3x - 18y = 9$$

$$\Rightarrow x - 6y = 3 \quad (3)$$

Solving (1) and (3), we get $x = -5, y = -\frac{4}{3}$.

Solving (2) and (3), we get $x = 3, y = 0$.

Thus, vertices of triangle are $(3, 2), (3, 0), (-5, -4/3)$.

$$\text{Area of triangle} = \frac{1}{2} \begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & -\frac{4}{3} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \times \left| 3 \left(\frac{4}{3} \right) - 2(3 + 5) + 1(-4) \right|$$

$$= \frac{1}{2} |4 - 16 - 4|$$

$$= 8 \text{ sq. units}$$

18. (1), (4)

The circle with points $P(2t_1, 2/t_1)$ and $Q(2t_2, 2/t_2)$ as diameter is given by

$$(x - 2t_1)(x - 2t_2) + \left(y - \frac{2}{t_1}\right)\left(y - \frac{2}{t_2}\right) = 1 \quad (1)$$

Also, the slope of PQ is given by

$$-\frac{1}{t_1 t_2} = 1 \text{ or } t_1 t_2 = -1$$

Hence, from (1), the circle is

$$(x^2 + y^2 - 8) - 2(t_1 + t_2)(x - y) = 0$$

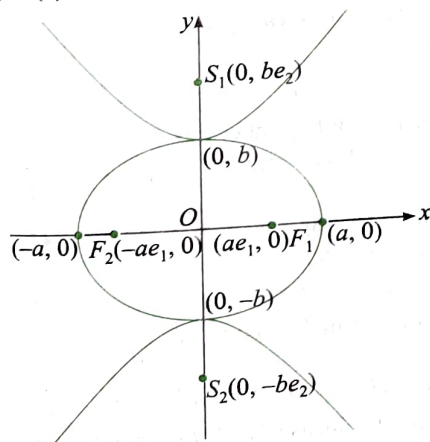
which is of the form $S + \lambda L = 0$.

Hence, circles pass through the points of intersection of the circle $x^2 + y^2 - 8 = 0$ and the line $x = y$.

The points of intersection are $(2, 2)$ and $(-2, -2)$.

Linked Comprehension Type

1. (2), 2. (3), 3. (4)



We have

$$b^2 = a^2(1 - e_1^2)$$

$$\text{and } 2be_2 = 2a \Rightarrow e_2 = \frac{a}{b}$$

$$\text{So, } \frac{1}{e_2^2} = 1 - e_1^2$$

$$\Rightarrow e_2^2(1 - e_1^2) = 1$$

Tangent at point $P(a \cos \theta, b \sin \theta)$ on the ellipse is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

It passes through $(0, be_2)$.

$$\text{So, } e_2 \sin \theta = 1$$

$$\Rightarrow \sin \theta = \frac{1}{e_2}$$

$$\therefore \theta = \tan^{-1} \left(\frac{1}{\sqrt{e_2^2 - 1}} \right)$$

$$A_1 = 4 \times \frac{1}{2} \times ae_1 \times be_2 = 2abe_1e_2$$

$$A_2 = \left(\frac{2a}{e_1} \right) \left(\frac{2b}{e_2} \right) = \frac{4ab}{e_1e_2}$$

$$\frac{A_1}{A_2} = \frac{e_1^2 e_2^2}{2} = 2$$

$$\Rightarrow e_1e_2 = 2$$

$$\text{But } e_2^2(1 - e_1^2) = 1$$

$$\text{So, } e_2^2 - 4 = 1$$

$$\therefore e_2 = \sqrt{5}$$

$$\text{and } e_1 = \frac{2}{\sqrt{5}}$$

$$\therefore e_2 : e_1 = 5 : 2$$

4. (2), 5. (3), 6. (1)

Ellipse and hyperbola are orthogonal and therefore, they are confocal.

$$\text{So, } a_h e_h = a_e e_e$$

$$\Rightarrow a^2 + 9 = 41 - 16$$

$$\Rightarrow 9 + a^2 = 25$$

$$\Rightarrow a^2 = 16$$

$$\text{Thus, hyperbola is } \frac{x^2}{9} - \frac{y^2}{16} = 1.$$

So, common tangents to the circle and hyperbola are $x = \pm 3$.

Director circle of hyperbola does not exist as $a < b$.

Director circle of circle is

$$x^2 + (y - 3)^2 = 18$$

$$\Rightarrow x^2 + y^2 - 6y - 9 = 0$$

This meets the hyperbola $16x^2 - 9y^2 = 144$ at four points from where tangents drawn to the circle $x^2 + (y - 3)^2 = 9$ are perpendicular to each other.

Let midpoint of AB be (h, k) .

$$\text{So, equation of line } AB \text{ is } hx + ky - 3(y + k) = h^2 + k^2 - 6k.$$

Since tangents at C and D intersect at the directrix, CD is the focal chord of hyperbola.

So, AB passes through focus of the hyperbola and that is $(\pm 5, 0)$.

Therefore, required locus is $x^2 + y^2 \pm 5x - 3y = 0$.

7. (4) Center $\equiv (1, 2)$

$$\text{Radius of auxiliary circle} = a = \sqrt{(2-1)^2 + (5-2)^2} = \sqrt{10}$$

$$2ae = \sqrt{8^2 + 8^2} = 8\sqrt{2} \text{ or } e = \frac{4}{\sqrt{5}}$$

$$b^2 = a^2 e^2 - a^2 = 32 - 10 = 22$$

$$\text{or } 2b = 2\sqrt{22}$$

8. (2) Directrix is perpendicular to the transverse axis. Let it be $x + y = k$.

Its distance from the center is a/e . Therefore,

$$\frac{|1 + 2 - k|}{\sqrt{2}} = \frac{5}{2\sqrt{2}} \text{ or } k = 3 + \frac{5}{2} = \frac{11}{2}$$

9. (3) The tangent is

$$y - 5 = -\frac{5-2}{6-5}(x-2)$$

$$\text{or } 3x + y = 11$$

The hyperbola is

$$(x-5)^2 + (y-6)^2 = \frac{16}{5} \times \frac{(2x+2y-11)^2}{8}$$

Solving, we get

$$(x-5)^2 + (5-3x)^2 = \frac{2}{5}(2x+22-6x-11)^2$$

$$\text{or } 5[10x^2 - 40x + 50] = 2(11-4x)^2$$

$$\text{or } 9x^2 - 12x + 4 = 0$$

$$\text{or } (3x-2)^2 = 0$$

$$\text{or } x = \frac{2}{3}, y = 9$$

10. (3) $2a = 3$

Distance between the foci (1, 2) and (5, 5) is 5. Therefore,

$$2ae = 5$$

$$\text{or } e = \frac{5}{3}$$

Now, if e' is eccentricity of the corresponding conjugate hyperbola, then

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\text{or } e' = \frac{5}{4}$$

11. (4) Director circle is the given by

$$(x-h)^2 + (y-k)^2 = a^2 - b^2$$

the where (h, k) is center, i.e., the midpoint of foci

$$\left(\frac{1+5}{2}, \frac{2+5}{2}\right) = \left(3, \frac{7}{2}\right)$$

$$b^2 = a^2(e^2 - 1) = \left(\frac{3}{2}\right)^2 \left\{ \left(\frac{5}{3}\right)^2 - 1 \right\} = 4$$

Therefore, the director circle is

$$(x-3)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{9}{4} - 4$$

$$\text{or } (x-3)^2 + \left(\frac{y-7}{2}\right)^2 = -\frac{7}{4}$$

This does not represent any real point.

12. (2) The slope of the transverse axis is $3/4$.

Therefore, the angle of rotation is

$$\theta = \tan^{-1} \frac{3}{4}$$

13. (1) The equation of tangent in parametric form is given by

$$\frac{x-1}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = \pm 3\sqrt{2}$$

$$\text{or } A \equiv (4, -2), B \equiv (-2, 4)$$

The equations of asymptotes (OA and OB) are given by

$$y + 2 = \frac{-2}{4}(x - 4)$$

$$\text{or } 2y + x = 0$$

$$\text{and } y - 4 = \frac{4}{-2}(x + 2)$$

$$\text{or } 2x + y = 0$$

Hence, the combined equation of asymptotes is

$$(2x + y)(x + 2y) = 0$$

$$\text{or } 2x^2 + 2y^2 + 5xy = 0$$

$$14. (3) m_{OA} = -\frac{1}{2}, m_{OB} = -2$$

$$\tan \theta = \left| \frac{-1/2 + 2}{1 + 1} \right| = \frac{3}{4}$$

$$\text{or } \sin \theta = \frac{3}{5}$$

$$\text{or } \theta = \sin^{-1} \left(\frac{3}{5} \right)$$

15. (2) Let the equation of the hyperbola be

$$2x^2 + 2y^2 + 5xy + \lambda = 0$$

It passes through (1, 1). Therefore,

$$2 + 2 + 5 + \lambda = 0$$

$$\text{or } \lambda = -9$$

So, the hyperbola is

$$2x^2 + 2y^2 + 5xy = 9$$

The equation of the tangent at $(-1, 7/2)$ is given by

$$2x(-1) + 2y\left(\frac{7}{2}\right) + 5 \frac{x(7/2) + (-1)y}{2} = 9$$

$$\text{or } 3x + 2y = 4$$

16. (2) Any point on the hyperbola $xy = 16$ is $(4t, 4/t)$.

Normal at this point is $y - 4/t = t^2(x - 4t)$.

If the normal passes through $P(h, k)$, then

$$k - \frac{4}{t} = t^2(h - 4t)$$

$$\text{or } 4t^4 - t^3h + tk - 4 = 0$$

This equation has roots t_1, t_2, t_3, t_4 which are parameters of the four feet of normals on the hyperbola. Therefore,

$$\sum t_i = \frac{h}{4}$$

$$\sum t_i t_j = 0$$

$$\sum t_i t_j t_k = -\frac{k}{4}$$

$$\text{and } t_1 t_2 t_3 t_4 = -1$$

$$\therefore \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{4}$$

$$\text{or } y_1 + y_2 + y_3 + y_4 = k$$

According to the question,

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{16} = k$$

Hence, the locus of (h, k) is

$$x^2 = 16y$$

17. (3) $x^2 = 16y$

The equation of tangent of P is

$$x \cdot 4t = \frac{16(y + t^2)}{2}$$

$$\text{or } 4tx = 8y + 8t^2$$

$$\text{or } tx = 2y + 2t^2$$

$$A \equiv (2t, 0), B \equiv (0, -t^2)$$

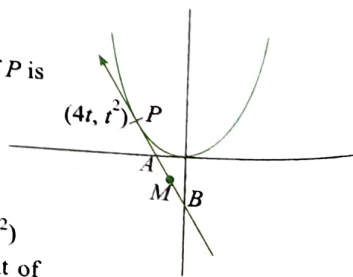
$M(h, k)$ is the middle point of AB.

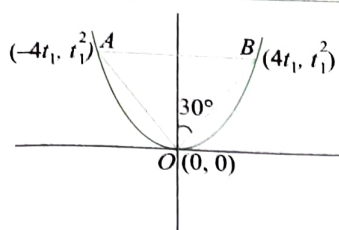
$$h = t, k = -\frac{t^2}{2} \text{ or } 2k = -h^2$$

Therefore, the locus of $M(h, k)$ is $x^2 + 2y = 0$.

18. (4) $\tan 30^\circ = \frac{4t_1}{t_1^2} = \frac{4}{t_1}$

$$\text{or } \frac{1}{\sqrt{3}} = \frac{4}{t_1} \text{ or } t_1 = 4\sqrt{3}$$





$$AB = 8t_1 = 32\sqrt{3}$$

$$\begin{aligned}\text{Area of } \triangle OAB &= \frac{\sqrt{3}}{4} \times 32\sqrt{3} \times 32\sqrt{3} \\ &= 768\sqrt{3} \text{ sq. units}\end{aligned}$$

19. (2) Perpendicular tangents intersect at the center of rectangular hyperbola. Hence, the center of the hyperbola is (1, 1) and the equations of asymptotes are $x - 1 = 0$ and $y - 1 = 0$.

20. (3) Let the equation of the hyperbola be $xy - x - y + 1 + \lambda = 0$.

It passes through (3, 2). Hence, $\lambda = -2$.

So, the equation of hyperbola is

$$xy = x + y + 1$$

21. (4) From the center of the hyperbola, we can draw two real tangents to the rectangular hyperbola.

Solving this with hyperbola, we have

$$x^2 - 16 = 9$$

$$\text{or } x^2 = 25 \text{ or } x = \pm 5$$

Hence, the points of contact are $(\pm 5, 4)$.

c. Obviously, the required point is $(-5, -4)$.

d. Let the points on the hyperbola be $P(h, k)$ and $Q(-h, k)$. Then

$$\text{Area of triangle} = \frac{1}{2} |2h| \cdot |k| = 10$$

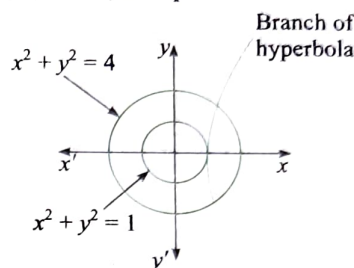
$$\text{or } h|6 + k| = 10$$

Also, points P and Q lie on the hyperbola. Hence,

$$h^2 - k^2 = 9$$

Clearly, points $(\pm 5, -4)$ satisfy both (1) and (2).

3. $a \rightarrow p$; $b \rightarrow s$; $c \rightarrow r$; $d \rightarrow p$



The locus of point P satisfying $PA - PB = 2$ is a branch of the hyperbola $x^2 - y^2/3 = 1$.

For $r = 2$, the circle and the branch of the hyperbola intersect at two points. For $r = 1$, there is one point of intersection.

If m is the slope of the common tangent, then

$$m^2 - 3 = r^2(1 + m^2)$$

$$\text{or } m^2 = \frac{r^2 + 3}{1 - r^2}$$

Hence, there are no common tangents for $r > 1$ and two common tangents for $r < 1$.

4. $a \rightarrow r$; $b \rightarrow p$; $c \rightarrow s$; $d \rightarrow q$

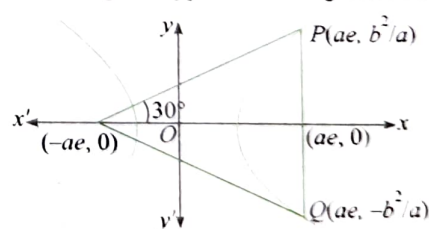
a. $\text{Im}(z^2) = 3$

$$\text{or } \text{Im}((x + iy)^2) = 3$$

$$\text{or } 2xy = 3$$

which is a rectangular hyperbola having eccentricity $\sqrt{2}$.

b.



$$\tan 30^\circ = \frac{b^2/a}{2ae}$$

$$\text{or } \frac{2}{\sqrt{3}} e = e^2 - 1$$

$$\text{or } \sqrt{3}e^2 - 2e - \sqrt{3} = 0$$

$$\text{or } e = \frac{2 \pm \sqrt{4 + 12}}{2\sqrt{3}} = \frac{2 \pm 4}{2\sqrt{3}}$$

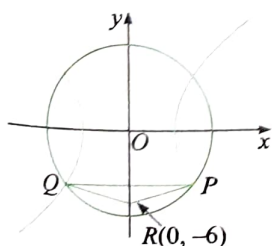
$$\text{or } e = \frac{3}{\sqrt{3}} = \sqrt{3}$$

c. Eccentricity of hyperbola = $\frac{AB}{PA - PB} = \frac{6}{4} = \frac{3}{2}$

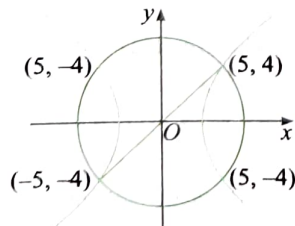
If the eccentricity of conjugate hyperbola is e' , then

$$\frac{1}{(3/2)^2} + \frac{1}{e'^2} = 1$$

$$\text{or } e' = \frac{3}{\sqrt{5}}$$



(a)



(b)

a. Obviously, all the points in List II are common to the hyperbola and circle.

b. The chord of contact of hyperbola w.r.t. $(0, -9/4)$ is

$$0(x) - \left(-\frac{9}{4}\right)y = 9$$

$$\text{or } y = 4$$

d. Angle between the asymptotes is

$$\tan^{-1} \left| \frac{2ab}{a^2 - b^2} \right| = \frac{\pi}{3}$$

$$\text{or } \left| \frac{2 \times a/b}{(a^2/b^2) - 1} \right| = \sqrt{3}$$

$$\text{or } \frac{2\sqrt{e'^2 - 1}}{|e'^2 - 2|} = \sqrt{3}$$

where e' is the eccentricity of conjugate hyperbola. Therefore, $e' = 2$.

5. (3) Focus of ellipse $\equiv (kae_1, 0)$

and

Focus of hyperbola $\equiv (ae_2, 0)$

Now, $kae_1 = ae_2$

$$\Rightarrow k \sqrt{1 - \frac{a^2}{k^2 a^2}} = \sqrt{1 + \frac{a^2}{a^2}}$$

$$\Rightarrow k^2 - 1 = 2$$

$$\therefore k = \pm \sqrt{3}$$

$$\text{a. } \frac{e_2}{e_1} = k = \sqrt{3}$$

b. Major axis of ellipse $= 2ka$

and

Transverse axis of hyperbola $= 2a$

$$\therefore \frac{2ka}{2a} = k = \sqrt{3}$$

c. Since curves are confocal, they are orthogonal.

$$\therefore \theta = \frac{\pi}{2}$$

d. Latus rectum of ellipse and hyperbola are $\frac{2a^2}{ak}$ and $\frac{2a^2}{a}$, respectively.

$$\therefore \frac{2a/k}{2a} = \frac{1}{k} = \frac{1}{\sqrt{3}}$$

Numerical Value Type

1. (5) For the given equation of the hyperbola, the foci are $S(3, 2)$ and $S'(-1, -1)$.

Using the definition of hyperbola, $|SP - S'P| = 2a$, we have $SS' = 5$ and $2a = 1$.

Hence, eccentricity is, $e = \frac{SS'}{2a} = 5$.

$$2. (5) e^2 = \frac{b^2}{a^2} + 1 \text{ or } \frac{b^2}{a^2} = e^2 - 1 = 24$$

Now, $y = mx + c$ is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then we must have

$$a^2 m^2 - b^2 \geq 0$$

$$\text{or } m^2 \geq b^2/a^2 \text{ or } m^2 \geq 24$$

then least positive integral value of m is 5.

3. (4) We have $y = Ax^2$, $y^2 + 3 = x^2 + 4y$; $A > 0$

$$\text{Now, } y^2 - 4y = x^2 - 3$$

$$\text{or } (y-2)^2 = x^2 + 1$$

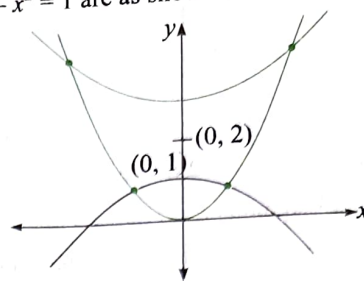
$$\text{or } (y-2)^2 - x^2 = 1$$

If $x = 0$, then

$$y - 2 = 1 \text{ or } -1$$

i.e., $y = 3$ or 1

Hence, the two graphs of $y = Ax^2$ ($A > 0$) and the hyperbola $(y-2)^2 - x^2 = 1$ are as shown which intersect at four points.



$$4. (9) \text{ Ellipse } \frac{(x-\sqrt{2})^2}{45/4} + \frac{(y-\sqrt{3})^2}{45/\lambda} = 1 \text{ and hyperbola}$$

$$\frac{(x-\sqrt{2})^2}{5} - \frac{(y-\sqrt{3})^2}{5/4} = 1 \text{ cut orthogonally.}$$

So, conics are confocal.

$$\therefore \frac{45}{4} - \frac{45}{\lambda} = 5 + \frac{5}{4}$$

$$\therefore \lambda = 9$$

5. (3) $(a, 2)$ lies on director circle $x^2 + y^2 = 7$. Therefore, $a^2 = 3$.

6. (7) The given hyperbola is

$$3x^2 - 2y^2 = 6$$

$$\text{or } \frac{x^2}{2} - \frac{y^2}{3} = 1$$

Equation of tangent is

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

$$\text{or } (y - mx)^2 = a^2 m^2 - b^2$$

Tangents from the point (α, β) will be

$$(\beta - m\alpha)^2 = 2m^2 - 3 \quad (\text{Since } a^2 = 2 \text{ and } b^2 = 3)$$

$$\text{or } m^2 \alpha^2 + \beta^2 - 2m\alpha\beta - 2m^2 + 3 = 0$$

$$m^2(\alpha^2 - 2) - 2\alpha\beta m + \beta^2 + 3 = 0$$

$$m_1 \cdot m_2 = \frac{\beta^2 + 3}{\alpha^2 - 2} = 2 = \tan \theta \tan \phi$$

$$\therefore \beta^2 + 3 = 2(\alpha^2 - 2)$$

$$\text{or } 2\alpha^2 - \beta^2 = 7$$

7. (8) The hyperbola is

$$x^2 - 9y^2 = 9$$

$$\text{or } \frac{x^2}{9} - \frac{y^2}{1} = 1$$

The equation of tangent is

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

It passes through $(3, 2)$. Therefore,

$$2 = 3m \pm \sqrt{9m^2 - 1}$$

$$\text{or } 4 + 9m^2 - 12m = 9m^2 - 1$$

Solving, we get the values of m as

$$m_1 = \frac{5}{12} \text{ and } m_2 = \infty$$

Therefore, equations of tangents are

$$5x - 12y + 9 = 0 \quad (2)$$

$$\text{and } x - 3 = 0 \quad (3)$$

Now, the equation of chord of contact w.r.t. point $P(3, 2)$ is $T = 0$

$$\text{or } 3x - 18y = 9$$

$$\text{or } x - 6y = 3 \quad (4)$$

Solving (2) and (4), we get

$$x = -5, y = -\frac{4}{3}$$

Solving (3) and (4), we get

$$x = 3, y = 0$$

Now, the vertices of triangle are $(3, 2)$, $(3, 0)$, and $(-5, -4/3)$.
Therefore,

$$\begin{aligned} \text{Area} &= \frac{1}{2} \begin{vmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ -5 & -4/3 & 1 \end{vmatrix} \\ &= 8 \text{ sq. units} \end{aligned}$$

8. (2) Equation of tangent to hyperbola $\frac{x^2}{5} - \frac{y^2}{b^2} = 1$ having slope m is

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

$$\Rightarrow y = x \pm \sqrt{5 - b^2}$$

Comparing with $y = x + 1$, we get

$$b^2 = 4 \text{ or } b = \pm 2$$

So, two values are possible.

9. (3) Since tangents drawn from the point $A(a, 2)$ are perpendicular, A must lie on the director circle $x^2 + y^2 = 7$. Putting $y = 2$, we get $x^2 = a^2 = 3$.

10. (6) Equations of director circles of ellipse and hyperbola are, respectively,

$$x^2 + y^2 = a^2 + b^2$$

$$\text{and } x^2 + y^2 = a^2 - b^2$$

According to the question, we have

$$a^2 + b^2 = 4r^2 \quad (1)$$

$$a^2 - b^2 = r^2 \quad (2)$$

Solving Eqs. (1) and (2), we get

$$a^2 = \frac{5r^2}{2}, b^2 = \frac{3r^2}{2}$$

$$e_1^2 = 1 - \frac{b^2}{a^2}$$

$$= 1 - \frac{3r^2}{2} \times \frac{2}{5r^2}$$

$$= 1 - \frac{3}{5} = \frac{2}{5}$$

$$e_2^2 = 1 + \frac{b^2}{a^2}$$

$$= 1 + \frac{3}{5} = \frac{8}{5}$$

$$\text{So, } 4e_2^2 - e_1^2 = 4 \times \frac{8}{5} - \frac{2}{5} = 6$$

11. (4) The equation of hyperbola is

$$(x - 3)(y - 2) = c^2$$

$$\text{or } xy - 2x - 3y + 6 = c^2$$

It passes through $(4, 6)$. Then,

$$4 \times 6 - 2 \times 4 - 3 \times 6 + 6 = c^2$$

$$\text{or } c = 2$$

$$\therefore \text{Latus rectum} = 2\sqrt{2} c = 2\sqrt{2} \times 2 = 4\sqrt{2}$$

12. (2) Let e and e' be the eccentricities of given hyperbola and its conjugate, respectively.

$$\text{Given that } 2 \tan^{-1} \left(\frac{b}{a} \right) = \frac{\pi}{3}$$

$$\Rightarrow \frac{b}{a} = \frac{1}{\sqrt{3}}$$

$$\text{So, } e^2 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\frac{1}{e'^2} + \frac{1}{e^2} = 1$$

$$\Rightarrow \frac{1}{e'^2} + \frac{3}{4} = 1$$

$$\Rightarrow \frac{1}{e'^2} = \frac{1}{4} \Rightarrow e' = 2$$

13. (24) The equation of hyperbola is

$$\frac{x^2}{16} - \frac{y^2}{18} = 1$$

$$\text{or } 9x^2 - 8y^2 - 144 = 0$$

Homogenizing this equation using

$$\frac{x \cos \alpha + y \sin \alpha}{p} = 1$$

$$\text{we have } 9x^2 - 8y^2 - 144 \left(\frac{x \cos \alpha + y \sin \alpha}{p} \right)^2 = 0$$

Since these lines are perpendicular to each other, we have

$$9p^2 - 8p^2 - 144(\cos^2 \alpha + \sin^2 \alpha) = 0$$

$$p^2 = 144 \text{ or } p = \pm 12$$

$$\therefore \text{Radius of circle} = 12$$

$$\therefore \text{Diameter of circle} = 24$$

Archives

JEE Main

Single Correct Answer Type

$$1. (2) \text{ Given } 2b = \frac{1}{2} (2ae) \Rightarrow b = \frac{ae}{2}$$

$$\text{Now } b^2 = a^2(e^2 - 1)$$

$$\Rightarrow a^2(e^2 - 1) = \frac{a^2 e^2}{4} \Rightarrow 3e^2 = 4 \Rightarrow e = \frac{2}{\sqrt{3}}$$

$$2. (3) \text{ Equation of hyperbola is } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$$\text{Foci are } (\pm 2, 0). \text{ So, } ae = 2.$$

$$\text{Now, } b^2 = a^2(e^2 - 1)$$

$$\therefore a^2 + b^2 = 4$$

$$\text{Given hyperbola passes through } (\sqrt{2}, \sqrt{3}).$$

$$\therefore \frac{2}{a^2} - \frac{3}{b^2} = 1$$

On solving (1) and (2), we get

$$a^2 = 1 \text{ and } b^2 = 3$$

So, equation of the hyperbola is $\frac{x^2}{1} - \frac{y^2}{3} = 1$.

Hence, equation of tangent at $P(\sqrt{2}, \sqrt{3})$ is $\frac{\sqrt{2}x}{1} - \frac{\sqrt{3}y}{3} = 1$,

which passes through the point $(2\sqrt{2}, 3\sqrt{3})$.

3. (2) In the figure, PQ is chord of contact w.r.t. point $T(0, 3)$.

Equation of PQ is

$$4(0 \times x) - (y \times 3) = 36 \text{ or } y = -12$$

Solving line PQ with the hyperbola, we get

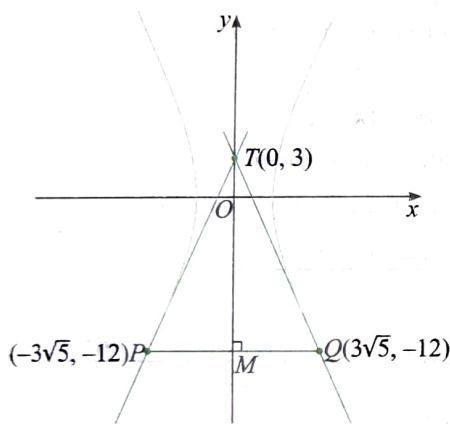
$$4x^2 - 144 = 36$$

$$\Rightarrow 4x^2 = 180$$

$$\Rightarrow x^2 = 45$$

$$\Rightarrow x = \pm 3\sqrt{5}$$

$$\therefore P(-3\sqrt{5}, -12), Q(3\sqrt{5}, -12)$$



$$\therefore \text{Area of triangle } TPQ = \frac{1}{2} \times PQ \times TM$$

$$= \frac{1}{2} \times 6\sqrt{5} \times 15 = 45\sqrt{5} \text{ sq. units}$$

Advanced

Multiple Correct Answer Type

$$1. (2) \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating w.r.t. x ,

$$\frac{dy}{dx} = \frac{xb^2}{ya^2}$$

Therefore, the slope of normal at $(6, 3)$ is $-a^2/2b^2$.

The equation of normal is

$$(y-3) = \frac{-a^2}{2b^2}(x-6)$$

It passes through the point $(9, 0)$. Therefore,

$$\frac{a^2}{2b^2} = 1 \text{ or } \frac{b^2}{a^2} = \frac{1}{2}. \quad \therefore e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{1}{2}$$

$$\therefore e = \sqrt{\frac{3}{2}}$$

Multiple Correct Answers Type

1. (1), (2)

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Since ellipse intersects hyperbola orthogonally. The ellipse and hyperbola will be confocal. So, comparing coordinates of foci

$$(\pm a \times \frac{1}{\sqrt{2}}, 0) \equiv (\pm 1, 0)$$

$$\text{or } a = \sqrt{2} \text{ and } e = \frac{1}{\sqrt{2}}$$

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$\text{or } b^2 = 1$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

2. (2), (4)

For the ellipse

$$\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1,$$

$$\text{we have } 1^2 = 2^2(1 - e^2)$$

$$\text{or } e = \frac{\sqrt{3}}{2}$$

Therefore, the eccentricity of the hyperbola is $2/\sqrt{3}$. So, for hyperbola

$$b^2 = a^2 \left(\frac{4}{3} - 1 \right)$$

$$\text{or } 3b^2 = a^2$$

One of the foci of the ellipse is $(\sqrt{3}, 0)$.

The hyperbola passes through $(\sqrt{3}, 0)$. Therefore, $\frac{3}{a^2} = 1$

$$\text{or } a^2 = 3 \text{ and } b^2 = 1$$

Therefore, the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{1} = 1$

$$\text{or } x^2 - 3y^2 = 3.$$

The focus of the hyperbola is

$$(ae, 0) \equiv \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0 \right) \equiv (2, 0)$$

3. (1), (2)

Slope of tangent = 2

The tangents are

$$y = 2x \pm \sqrt{9 \times 4 - 4} \quad (\text{using } y = mx \pm \sqrt{a^2m^2 - b^2})$$

$$\text{i.e., } 2x - y = \pm 4\sqrt{2}$$

$$\text{or } \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1 \text{ and } \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = -1$$

Comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$ (Eqn. of tangent to hyperbola

at point (x_1, y_1) on it) we get the points of contact as $(9/2\sqrt{2}, 1/\sqrt{2})$ and $(-9/2\sqrt{2}, -1/\sqrt{2})$.

Alternate Method:

The equation of tangent at $P(\theta)$ is

$$\left(\frac{\sec \theta}{3} \right) x - \left(\frac{\tan \theta}{2} \right) y = 1$$

$$\therefore \text{Slope} = \frac{2 \sec \theta}{3 \tan \theta} = 2$$

$$\text{or } \sin \theta = \frac{1}{3}$$

$$\therefore \sec \theta = \pm \frac{3}{2\sqrt{2}} \text{ and corresponding by } \tan \theta = \pm \frac{1}{2\sqrt{2}}$$

Therefore, the points are $(9/2\sqrt{2}, 1/\sqrt{2})$ and $(-9/2\sqrt{2}, -1/\sqrt{2})$.

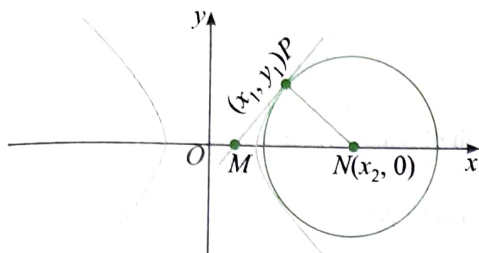
4. (1), (2), (4)

As shown in figure, hyperbola and circle touch at $P(x_1, y_1)$.

Equation of tangent to H at P is $xx_1 - yy_1 = 1$.

It meets the x -axis at $M(1/x_1, 0)$.

Now, centroid of $\triangle PMN$ is (l, m) .



$$\text{So, } l = \frac{x_1 + x_2 + \frac{1}{x_1}}{3} \text{ and } m = \frac{y_1}{3} = \frac{\sqrt{x_1^2 - 1}}{3}$$

$$\text{Now, } \left. \frac{dy}{dx} \right|_{H \text{ at } P} = \left. \frac{dy}{dx} \right|_{S \text{ at } P}$$

$$\Rightarrow \frac{x_1}{y_1} = \frac{x_2 - x_1}{y_1}$$

$$\Rightarrow x_2 = 2x_1$$

$$\text{So, } l = x_1 + \frac{1}{3x_1}$$

$$\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \frac{dm}{dy_1} = \frac{1}{3}, \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

5. (2), (3), (4)

The line $y = mx + c$ is tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 - b^2$.

Given tangent is: $y = 2x + 1$

$$\therefore (1)^2 = 4a^2 - 16$$

$$\Rightarrow a^2 = \frac{17}{4}$$

$$\Rightarrow a = \frac{\sqrt{17}}{2}$$

For option (1), sides are $\sqrt{17}, 4, 1$, which form right angled triangle.

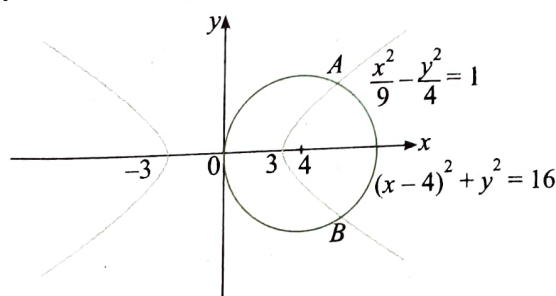
For option (2), sides are $\sqrt{17}, 8, 1$, for which triangle is not possible.

For option (3), sides are $\frac{\sqrt{17}}{2}, 4, 1$, for which triangle is not possible.

For option (4), sides are $\frac{\sqrt{17}}{2}, 4, 2$, for which triangle exists but not right angled.

Linked Comprehension Type

(2)



A tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$, having slope m , is

$$y = mx + \sqrt{9m^2 - 4}, m > 0$$

It is tangent to $x^2 + y^2 - 8x = 0$. Therefore, its distance from the centre of the circle is equal to the radius of circle.

$$\therefore \frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1 + m^2}} = 4$$

$$\text{or } 495m^4 + 104m^2 - 400 = 0$$

$$\text{or } m^2 = \frac{4}{5} \text{ or } m = \frac{2}{\sqrt{5}}$$

Therefore, the tangent is

$$y = \frac{2}{\sqrt{5}}x + \frac{4}{\sqrt{5}}$$

$$\text{or } 2x - \sqrt{5}y + 4 = 0$$

2. (1) A point on the hyperbola is $(3 \sec \theta, 2 \tan \theta)$.

It lies on the circle. So, $9 \sec^2 \theta + 4 \tan^2 \theta - 24 \sec \theta = 0$,

$$\text{i.e., } 13 \sec^2 \theta - 24 \sec \theta - 4 = 0$$

$$\text{or } \sec \theta = 2, -\frac{2}{13}$$

Clearly circle cuts the hyperbola in first and fourth quadrants.

$$\therefore \sec \theta = 2, \tan \theta = \sqrt{3}$$

The points of intersection are $A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$.

Therefore, the circle with AB as diameter is

$$(x - 6)^2 + y^2 = (2\sqrt{3})^2$$

$$\text{or } x^2 + y^2 - 12x + 24 = 0$$

Matrix Match Type

1. d $\rightarrow$ q, s.

q. If $|z - z_1| - |z - z_2| = k$ where $k < |z_1 - z_2|$, then locus of variable point 'z' is on branch of the hyperbola with fixed points z_1 and z_2 foci.

$$\text{Given } |z + 2| - |z - 2| = 3$$

Clearly distance between complex numbers '2' and '-2' is 4 which is less than 3.

So, locus of z is a branch of the hyperbola.

s. If eccentricity is $[1, \infty)$, then the conic can be a parabola (if $e = 1$) and a hyperbola if $e \in (1, \infty)$.

Note: Solutions of the remaining parts are given in their respective chapters.

2. (4) Tangent at $\left(\sqrt{3}, \frac{1}{2}\right)$ is

$$\sqrt{3}x + 2y = 4 \quad (1)$$

Since slope of tangent at $\left(\sqrt{3}, \frac{1}{2}\right)$ is -ve, possible curves are

(I) and (II) only (draw the diagram and verify).

Also, given equation of tangent cannot match with

$$my = m^2x + a.$$

So, comparing eq. (1) with $y = mx + a\sqrt{m^2 + 1}$, we get

$$a = 2 \text{ and } m = \frac{-\sqrt{3}}{2}.$$

Therefore, equation of curve is $\frac{x^2}{4} + y^2 = 1$.

The corresponding point of contact is

$$\left(\frac{-a^2m}{\sqrt{a^2m^2+1}}, \frac{1}{\sqrt{a^2m^2+1}} \right).$$

3. (1) Tangent at (8, 16) is $y = x + 8$

Since the slope of tangent is +ve, possible curve will be $y^2 = 4ax$.

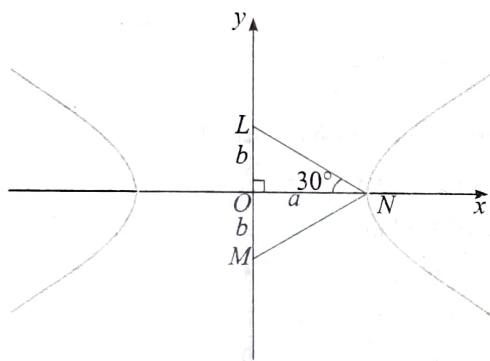
Hence, equation of tangent is $my = m^2x + a$ and point of contact is $\left(\frac{a}{m^2}, \frac{2a}{m} \right)$.

4. (4) For $a = \sqrt{2}$ and point $(-1, 1)$ on the curve, the only possible curve is $x^2 + y^2 = a^2$.

Equation of tangent is $y = mx + a\sqrt{m^2+1}$ and point of contact

is $\left(\frac{-ma}{\sqrt{m^2+1}}, \frac{a}{\sqrt{m^2+1}} \right)$.

5. (2)



Area of $\triangle LMN = 4\sqrt{3}$

$$\frac{1}{2}(2b)(\sqrt{3}b) = 4\sqrt{3}$$

$$\Rightarrow b^2 = 4 \Rightarrow b = 2 \Rightarrow 2b = 4$$

$$\text{Here, } \frac{a}{b} = \cot 30^\circ \Rightarrow a = \sqrt{3}b \Rightarrow a = 2\sqrt{3}$$

$$\text{Now, } b^2 = a^2(e^2 - 1)$$

$$\therefore 4 = 12(e^2 - 1)$$

$$\Rightarrow e^2 = 1 + \frac{1}{3} = \frac{4}{3}$$

$$\Rightarrow e = \frac{2}{\sqrt{3}}$$

$$\text{Distance between foci} = 2ae = 2 \times 2\sqrt{3} \times \frac{2}{\sqrt{3}} = 8$$

$$\text{Length of latus rectum} = \frac{2b^2}{a} = \frac{2 \times 4}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Numerical Value Type

1. (2) Substituting $(a/e, 0)$ in $y = -2x + 1$, we get

$$0 = -\frac{2a}{e} + 1$$

$$\text{or } \frac{2a}{e} = 1$$

$$\text{or } a = \frac{e}{2}$$

$$\text{Also, } 1 = \sqrt{a^2m^2 - b^2}$$

$$\text{or } 1 = a^2m^2 - b^2$$

$$\text{or } 1 = 4a^2 - b^2$$

$$\text{or } 1 = \frac{4e^2}{4} - b^2$$

$$\text{or } b^2 = e^2 - 1$$

$$\text{Also, } b^2 = a^2(e^2 - 1)$$

$$\therefore a = 1, e = 2$$

Chapterwise Solved January 2019 JEE Main Questions (All Sets)

Chapter 1 (Coordinate System)

- Two vertices of a triangle are $(0, 2)$ and $(4, 3)$. If its orthocentre is at the origin, then its third vertex lies in which quadrant?
(1) Fourth (2) Second (3) Third (4) First
- If the straight line $2x - 3y + 17 = 0$ is perpendicular to the line passing through the points $(7, 17)$ and $(15, \beta)$, then β equals
(1) -5 (2) $-\frac{35}{3}$ (3) $\frac{35}{3}$ (4) 5

Chapter 2 (Straight Lines)

- Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true?
(1) The lines are all parallel.
(2) Each line passes through the origin.
(3) The lines are not concurrent.
(4) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$.
- Let the equations of two sides of a triangle be $3x - 2y + 6 = 0$ and $4x + 5y - 20 = 0$. If the orthocentre of this triangle is at $(1, 1)$, then the equation of its third side is
(1) $122y - 26x - 1675 = 0$ (2) $26x + 61y + 1675 = 0$
(3) $122y + 26x + 1675 = 0$ (4) $26x - 122y - 1675 = 0$
- If the line $3x + 4y - 24 = 0$ intersects the x -axis at the point A and the y -axis at the point B , then the incentre of the triangle OAB , where O is the origin, is
(1) $(3, 4)$ (2) $(2, 2)$ (3) $(4, 4)$ (4) $(4, 3)$
- A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of ΔPQR is a line
(1) parallel to x -axis (2) with slope $\frac{2}{3}$
(3) with slope $\frac{3}{2}$ (4) parallel to y -axis

- Two sides of a parallelogram are along the lines $x + y = 3$ and $x - y + 3 = 0$. If its diagonals intersect at $(2, 4)$, then one of its vertices is
(1) $(2, 6)$ (2) $(2, 1)$ (3) $(3, 5)$ (4) $(3, 6)$
- If in a parallelogram $ABCD$, the coordinates of A, B and C are, respectively, $(1, 2)$, $(3, 4)$ and $(2, 5)$, then the equation of the diagonal AD is
(1) $5x + 3y - 11 = 0$ (2) $3x - 5y + 7 = 0$
(3) $3x + 5y - 13 = 0$ (4) $5x - 3y + 1 = 0$
- If a straight line passing through the point $P(-3, 4)$ is such that its intercepted portion between the coordinate axes is bisected at P , then its equation is
(1) $x - y + 7 = 0$ (2) $3x - 4y + 25 = 0$
(3) $4x + 3y = 0$ (4) $4x - 3y + 24 = 0$
- The tangent to the curve $y = x^2 - 5x + 5$, parallel to the line $2y = 4x + 1$, also passes through the point
(1) $\left(\frac{1}{4}, \frac{7}{2}\right)$ (2) $\left(\frac{7}{2}, \frac{1}{4}\right)$
(3) $\left(-\frac{1}{8}, 7\right)$ (4) $\left(\frac{1}{8}, -7\right)$

Chapter 4 (Circle)

- Three circles of radii a, b, c ($a < b < c$) touch each other externally. If they have x -axis as a common tangent, then
(1) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$ (2) a, b, c are in A.P.
(3) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in A.P. (4) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$
- If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x - 4)^2 + (y - 7)^2 = 36$ intersect at two distinct points, then
(1) $0 < r < 1$ (2) $1 < r < 11$ (3) $r > 11$ (4) $r = 11$
- If a circle C passing through the point $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point $(1, -1)$, then the radius of C is
(1) $\sqrt{57}$ (2) 4 (3) $2\sqrt{5}$ (4) 5

14. A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the coordinate axes. Then the distance of the vertex of this square which is nearest to the origin is
 (1) 13 (2) $\sqrt{137}$ (3) 6 (4) $\sqrt{41}$
15. The straight line $x + 2y = 1$ meets the coordinate axes at A and B . A circle is drawn through A , B and the origin. Then the sum of perpendicular distances from A and B on the tangent to the circle at the origin is
 (1) $\frac{\sqrt{5}}{4}$ (2) $\frac{\sqrt{5}}{2}$ (3) $2\sqrt{5}$ (4) $24\sqrt{5}$
16. Two circles with equal radii are intersecting at the points $(0, 1)$ and $(0, -1)$. The tangent at the point $(0, 1)$ to one of the circles passes through the centre of the other circle. Then the distance between the centres of these circles is
 (1) 1 (2) $\sqrt{2}$ (3) $2\sqrt{2}$ (4) 2
17. A circle cuts a chord of length $4a$ on the x -axis and passes through a point on the y -axis, distant $2b$ from the origin. Then the locus of the centre of this circle is
 (1) A hyperbola (2) A parabola
 (3) A straight line (4) An ellipse
18. If a variable line $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval
 (1) $[12, 21]$ (2) $(2, 17)$
 (3) $(23, 31)$ (4) $[13, 23]$
19. Let C_1 and C_2 be the centres of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 6x - 6y + 14 = 0$, respectively. If P and Q are the points of intersection of these circles, then the area (in sq. units) of the quadrilateral PC_1QC_2 is
 (1) 8 (2) 6 (3) 9 (4) 4
20. If a circle of radius R passes through the origin O and intersects the coordinate axes at A and B , then the locus of the foot of perpendicular from O on AB is
 (1) $(x^2 + y^2)^2 = 4R^2 x^2 y^2$
 (2) $(x^2 + y^2)(x + y) = R^2 xy$
 (3) $(x^2 + y^2)^3 = 4R^2 x^2 y^2$
 (4) $(x^2 + y^2)^2 = 4R^2 x^2 y^2$
21. Equation of a common tangent to the circle $x^2 + y^2 - 6x = 0$ and the parabola $y^2 = 4x$ is
 (1) $2\sqrt{3}y = 12x + 1$ (2) $2\sqrt{3}y = -x - 12$
 (3) $\sqrt{3}y = x + 3$ (4) $\sqrt{3}y = 3x + 1$
22. Axis of a parabola lies along x -axis. If its vertex and focus are at distances 2 and 4, respectively, from the origin on the positive x -axis, then which of the following points does not lie on it?
 (1) $(4, -4)$ (2) $(5, 2\sqrt{6})$
 (3) $(8, 6)$ (4) $(6, 4\sqrt{2})$
23. Let $A(4, -4)$ and $B(9, 6)$ be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB is
 (1) $31\frac{3}{4}$ (2) 32 (3) $30\frac{1}{2}$ (4) $31\frac{1}{4}$
24. If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal other than x -axis, then which one of the following is a valid choice for the ordered triad (a, b, c) ?
 (1) $(1, 1, 0)$ (2) $(\frac{1}{2}, 2, 3)$
 (3) $(\frac{1}{2}, 2, 0)$ (4) $(1, 1, 3)$
25. The length of the chord of the parabola $x^2 = 4y$ having equation $x - \sqrt{2}y + 4\sqrt{2} = 0$ is
 (1) $2\sqrt{11}$ (2) $3\sqrt{2}$ (3) $6\sqrt{3}$ (4) $8\sqrt{2}$
26. If the area of the triangle whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and y -axis, is 250 sq. units, then a value of a is
 (1) $5\sqrt{5}$ (2) $(10)^{2/3}$ (3) $5(2^{1/3})$ (4) 5
27. Let $P(4, -4)$ and $Q(9, 6)$ be two points on the parabola, $y^2 = 4x$ and let X be any point on the arc POQ of this parabola, where O is the vertex of this parabola, such that the area of ΔPXQ is maximum. Then this maximum area (in sq. units) is
 (1) $\frac{125}{4}$ (2) $\frac{125}{2}$ (3) $\frac{625}{4}$ (4) $\frac{75}{2}$
28. The equation of a tangent to the parabola, $x^2 = 8y$, which makes an angle θ with the positive direction of x -axis, is
 (1) $x = y \cot \theta + 2 \tan \theta$ (2) $x = y \cot \theta - 2 \tan \theta$
 (3) $y = x \tan \theta - 2 \cot \theta$ (4) $y = x \tan \theta + 2 \cot \theta$

29. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices, then the midpoints of the tangents intercepted between the coordinate axes lie on the curve

(1) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (2) $\frac{x^2}{4} + \frac{y^2}{2} = 1$
(3) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ (4) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$

30. Let the length of the latus rectum of an ellipse with its major axis along x -axis and centre at the origin be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it?

(1) $(4\sqrt{3}, 2\sqrt{3})$ (2) $(4\sqrt{3}, 2\sqrt{2})$
(3) $(4\sqrt{2}, 2\sqrt{2})$ (4) $(4\sqrt{2}, 2\sqrt{3})$

31. Let S and S' be the foci of the ellipse and B be any one of the extremities of its minor axis. If $\Delta S'BS = 8$ sq. units, then the length of a latus rectum of the ellipse is

(1) $2\sqrt{2}$ (2) 2 (3) 4 (4) $4\sqrt{2}$

Chapter 7
(Hyperbola)

32. Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2 \theta} - \frac{y^2}{\sin^2 \theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval

(1) (2, 3] (2) (3, ∞) (3) $(3/2, 2]$ (4) $(1, 3/2]$

33. A hyperbola has its centre at the origin, passes through the point $(4, 2)$ and has transverse axis of length 4 along the x -axis. Then the eccentricity of the hyperbola is

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{3}{2}$ (3) $\sqrt{3}$ (4) 2

34. The equation of a tangent to the hyperbola $4x^2 - 5y^2 = 20$ parallel to the line $x - y = 2$ is
(1) $x - y + 9 = 0$ (2) $x - y + 7 = 0$
(3) $x - y + 1 = 0$ (4) $x - y - 3 = 0$

35. Let $S = \left\{ (x, y) \in \mathbb{R}^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1 \right\}$, where $r \neq \pm 1$. Then S represents

(1) A hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, where $0 < r < 1$.

(2) An ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, where $r > 1$.

(3) A hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, where $0 < r < 1$.

(4) An ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, where $r > 1$.

36. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbola $xy = 2$ is

(1) $x + 2y + 4 = 0$ (2) $x - 2y + 4 = 0$
(3) $x + y + 1 = 0$ (4) $4x + 2y + 1 = 0$

37. If a hyperbola has length of its conjugate axis equal to 5 and the distance between its foci is 13, then the eccentricity of the hyperbola is

(1) 2 (2) $\frac{13}{6}$ (3) $\frac{13}{8}$ (4) $\frac{13}{12}$

38. If the vertices of a hyperbola be at $(-2, 0)$ and $(2, 0)$ and one of its foci be at $(-3, 0)$, then which one of the following points does not lie on this hyperbola?

(1) $(4, \sqrt{15})$ (2) $(-6, 2\sqrt{10})$
(3) $(6, 5\sqrt{2})$ (4) $(2\sqrt{6}, 5)$

Answers Key

- | | | | | | | | | | |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 1. (2) | 2. (4) | 3. (4) | 4. (4) | 5. (2) | 21. (3) | 22. (3) | 23. (4) | 24. (4) | 25. (3) |
| 6. (2) | 7. (4) | 8. (4) | 9. (4) | 10. (4) | 26. (4) | 27. (1) | 28. (1) | 29. (3) | 30. (2) |
| 11. (1) | 12. (2) | 13. (4) | 14. (4) | 15. (2) | 31. (3) | 32. (2) | 33. (1) | 34. (3) | 35. (4) |
| 16. (4) | 17. (2) | 18. (1) | 19. (4) | 20. (3) | 36. (1) | 37. (4) | 38. (2) | | |